Systems of difference and differential equations

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Structured Populations

Not all individuals in a population are equal and have the same chances of survival and reproduction.

Demographic characteristics are dependent on population structure. Individuals differ in many ways and some of these differences result in variations in fertility and survival rates.

Classical demographic analysis is based on an age-specific survival and reproduction “tabulation” system known as the **life table**.

The basic information needed to study changes in density and growth or decrease rates is contained in the life table.

The most usual model to study this populations are the **Leslie Matrix**.

Structured Populations

We consider:

- there are an equal number of males and females
- the life limit of the species is $\boldsymbol{m}$ years
- the incubation time is $\boldsymbol{d}$ years
- the population is fertile until death
- $\boldsymbol{n}$ is the age group to which a given female belongs
- $\boldsymbol{x_n(t)}$ is number of females in age $\boldsymbol{n}$ months at the beginning of year $\boldsymbol{t}$
- $\boldsymbol{s_n}$ is the survival rate in age group $\boldsymbol{n}$
- $\boldsymbol{f_n}$ is the fertility rate in age group $\boldsymbol{n}$

Leslie Matrix

The Leslie matrix is given by

$$L = \begin{bmatrix} f_0 & f_1 & f_2 & \cdots & f_{m-1} & f_m \\ s_0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & s_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & s_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \cdots \\ 0 & 0 & 0 & \cdots & s_{m-1} & 0 \end{bmatrix}$$

To find the number of individuals in each age group we have

$$\begin{bmatrix} x_0(t) \\ x_1(t) \\ x_2(t) \\ \vdots \\ x_m(t) \end{bmatrix} = L \begin{bmatrix} x_0(t-1) \\ x_1(t-1) \\ x_2(t-1) \\ \vdots \\ x_m(t-1) \end{bmatrix} = L^{t-1} \begin{bmatrix} x_0(0) \\ x_1(0) \\ x_2(0) \\ \vdots \\ x_m(0) \end{bmatrix} \Longrightarrow X(t) = L^{t-1} X(0)$$

The growth rate is given for the largest absolute eigenvalue of the matrix L, i.e,

is the largest absolute root of the characteristic equation

$$\lambda = f_0 + s_0 f_1 \lambda^{-1} + s_0 s_1 f_2 \lambda^{-2} + \cdots + s_0 s_1 \ldots s_{m-1} f_m \lambda^{-m}$$

Leslie Matrix

Consider an elephant's community in Botswana

such that:

- the life limit of the species is **60** years
- the incubation time is **2** years
- $s_0 = s_1 = s_2 = 0{,}9 \qquad s_3 = s_4 = \dots = s_{50} = 0{,}965$

  $s_{51} = s_{52} = \dots = s_{60} = 0{,}8$
- $f_1 = f_2 = \dots = f_9 = 0 \qquad f_{10} = f_{11} = f_{12} = 0{,}12 \qquad f_{13} = \dots = f_{60} = 0{,}21$

Leslie Matrix

The characteristic equation is given by

$$\lambda \approx 0{,}06811\lambda^{-10} + 0{,}06579\lambda^{-11} + 0{,}06348\lambda^{-12} + 0{,}10721\lambda^{-13} + 0{,}10345\lambda^{-14}$$

$$+0{,}09983\lambda^{-15} + 0{,}09634\lambda^{-16} + 0{,}09297\lambda^{-17} + 0{,}08971\lambda^{-18} + 0{,}08657\lambda^{-1}$$

$$+0{,}08354\lambda^{-20} + 0{,}08062\lambda^{-21} + 0{,}0778\lambda^{-22} + 0{,}07507\lambda^{-23} + 0{,}07245\lambda^{-24}$$

$$+0{,}06991\lambda^{-25} + 0{,}06746\lambda^{-26} + 0{,}0651\lambda^{-27} + 0{,}06282\lambda^{-2} + 0{,}06063\lambda^{-2}$$

$$+0{,}0585\lambda^{-30} + 0{,}05646\lambda^{-31} + 0{,}05448\lambda^{-32} + 0{,}05257\lambda^{-3} + 0{,}05073\lambda^{-3}$$

$$+0{,}04896\lambda^{-35} + 0{,}04724\lambda^{-36} + 0{,}04559\lambda^{-3} + 0{,}04399\lambda^{-3} + 0{,}04246\lambda^{-3}$$

$$+0{,}04097\lambda^{-40} + 0{,}03954\lambda^{-4} + 0{,}03815\lambda^{-4} + 0{,}03682\lambda^{-43} + 0{,}03553\lambda^{-44}$$

$$+0{,}03428\lambda^{-45} + 0{,}03308\lambda^{-46} + 0{,}03193\lambda^{-4} + 0{,}03081\lambda^{-48} + 0{,}02973\lambda^{-49}$$

$$+0{,}02869\lambda^{-50} + 0{,}02769\lambda^{-51} + 0{,}02215\lambda^{-52} + 0{,}01772\lambda^{-53} + 0{,}01418\lambda^{-54}$$

$$+0{,}01134\lambda^{-55} + 0{,}00907\lambda^{-56} + 0{,}00726\lambda^{-57} + 0{,}00581\lambda^{-58} + 0{,}00464\lambda^{-59}$$

$$+0{,}00372\lambda^{-60}$$

Leslie Matrix

The biggest root of the characteristic solution is $\boldsymbol{\lambda \approx 1,0376}$.

This is a grow of $\boldsymbol{3,76\%}$ by year.

If we change the survival rates from

$\boldsymbol{s_0 = s_1 = s_2 =} 0{,}9 \qquad \boldsymbol{s_3 = s_4 = \ldots = s_{50} =} 0{,}965$

$\boldsymbol{s_{51} = s_{52} = \ldots = s_{60} =} 0{,}8$

to

$\boldsymbol{s_0 = s_1 = s_2 =} 0{,}9 \qquad \boldsymbol{s_3 = s_4 = \ldots = s_9 =} 0{,}9$

$\boldsymbol{s_{10} = \cdots = s_{50} =} 0{,}8 \qquad \boldsymbol{s_{51} = \cdots = s_{60} =} 0{,}7$

we obtain $\boldsymbol{\lambda \approx 0,955}$. **Extinction!**

Interactions between species

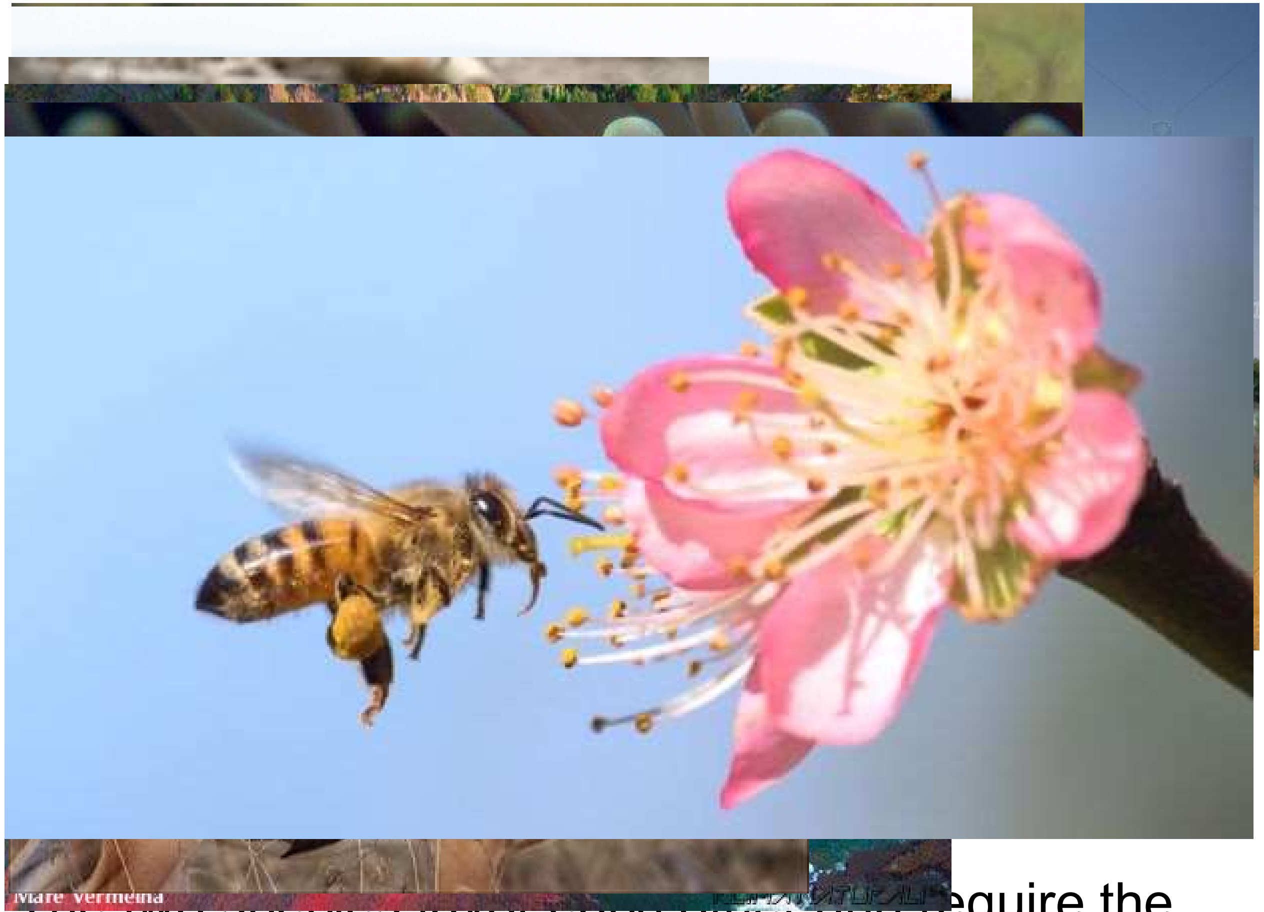
- Neutralism — Neither species affects the other.

Negative Interactions

- Direct Competition
- Indirect Competition
- Amensalism
- Parasitism
- Predation

Positive interactions

- Commensalism
- Protocooperation
- Mutualism



The two species favor each other and require the presence of the other.

Predator-prey models

- $y(t)$ the number of predators at a given time t
- $x(t)$ the number of prey at that time.

We consider:

- the prey population $x(t)$ is the total supply of food available to predators,
- the total food consumed by predators is proportional to the number of predator-prey encounters.

So, ~~ignoring for now social phenomena~~, we get the equations:

Predator-prey
Lotka-Volterra equations

$$\begin{cases} x' = Ax - Bxy \\ y' = -Dy + Cxy \end{cases}$$

Social phenomena

where $\mathrm{A}, \mathrm{B}, C$ and D are positive constants.

Predator-prey models

Equilibrium points are given by

$$\begin{cases} x' = 0 \\ y' = 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} Ax - Bxy - \lambda x^2 = 0 \\ -Dy + Cxy - \mu y^2 = 0 \end{cases}$$

So

$$x = 0 \text{ and } y = 0,$$

or

$$A - By - \lambda x = 0 \text{ and } y = 0, \qquad (x = A/\lambda \text{ and } y = 0)$$

or

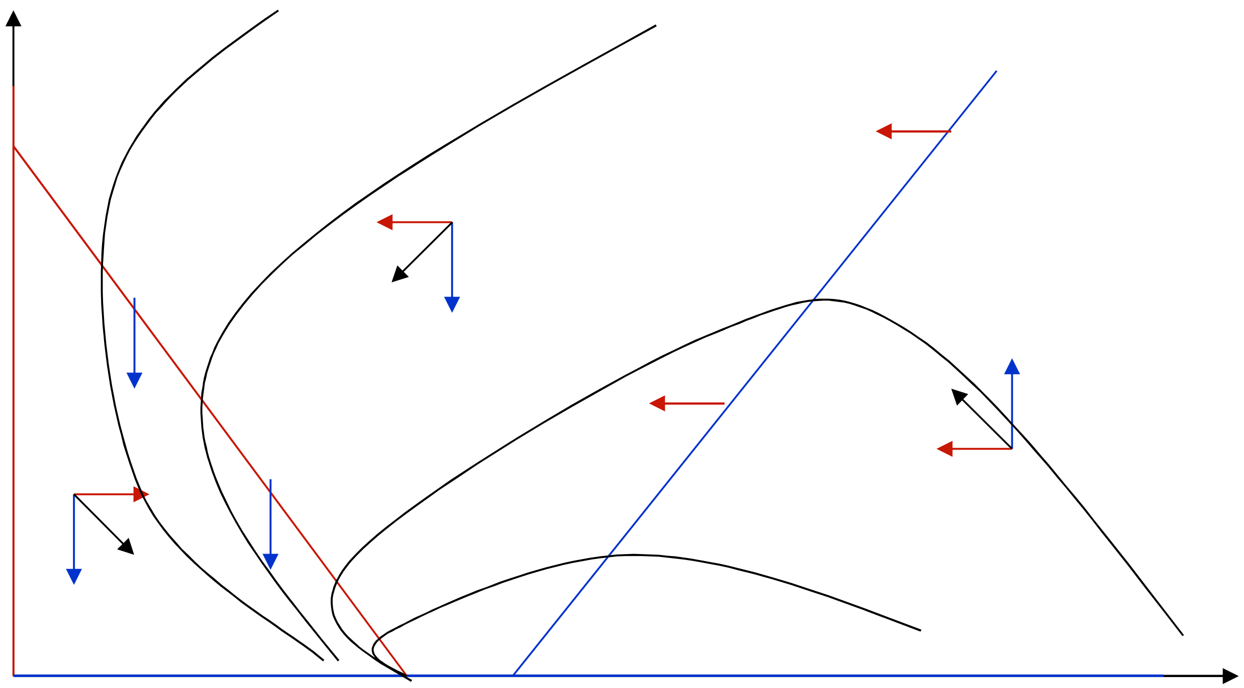
$$x = 0 \text{ and } Cx - D - \mu y = 0, \text{ (impossible because } y > 0)$$

or

$$A - By - \lambda x = 0 \quad \text{and} \quad Cx - D - \lambda y = 0.$$

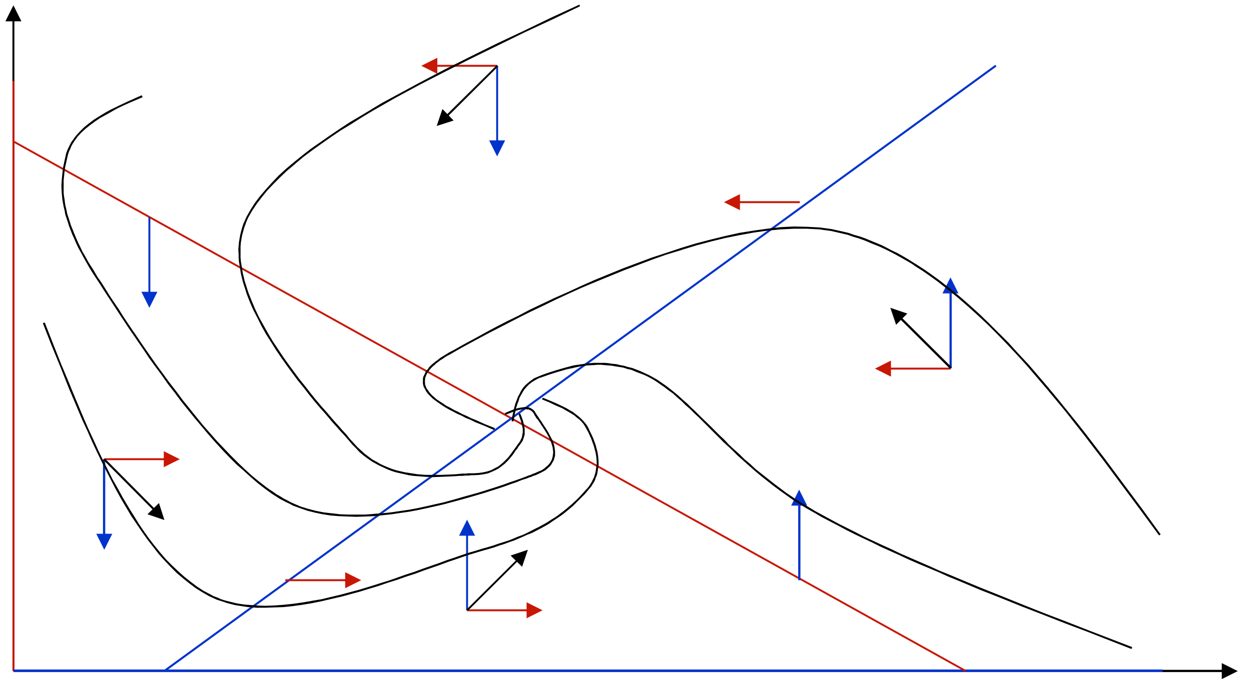
Predator-prey models

The lines $A - By - \lambda x = 0$ and $Cx - D - \mu y = 0$ do not intersect.



Predator-prey models

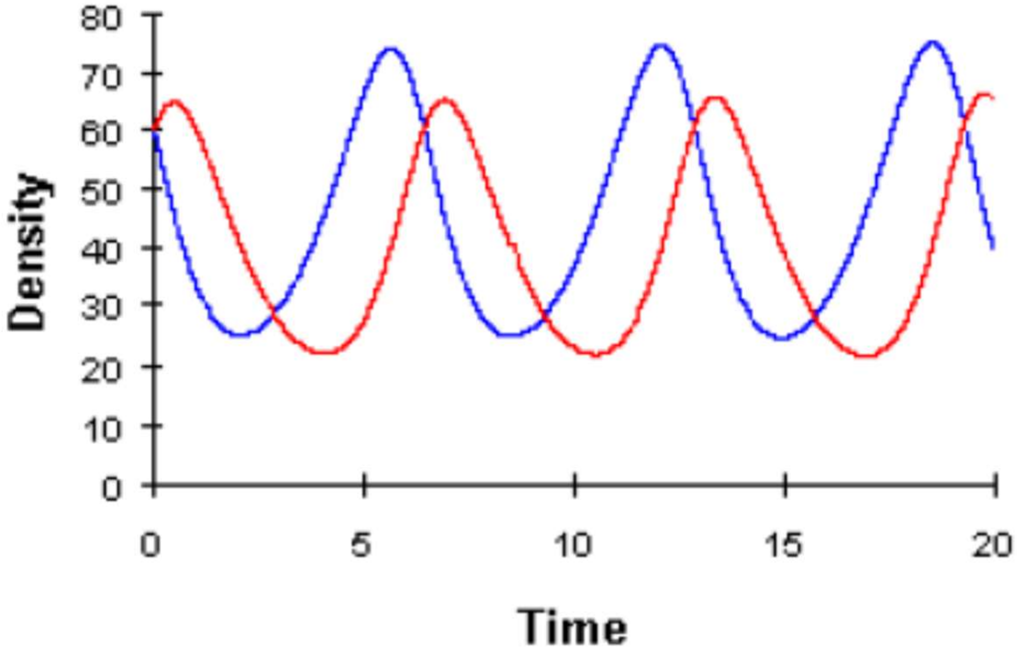
The lines $A - By - \lambda x = 0$ and $Cx - D - \mu y = 0$ intersect.



Predator-prey models

Oscillation of prey and predator densities according to the Lotka-Volterra

Model: The prey is in blue and its highs precede those of the predator (in red).



Oscillatory behavior about the equilibrium point

Competition models

There is competition between organisms whenever one has a negative effect on another, either by consuming or controlling access to a resource whose availability is limited.

There are two ways in which an organism uses space to consume its fair share of available resources:

- **Collectivist** - individuals move more or less freely throughout the area, gathering resources as they move. In this case, individuals are only harmed by consuming resources that would otherwise be available to others. The negative effects are therefore indirect.
- **Monopolist** - An essential resource is obtained by occupying a portion of space more or less exclusively.

Competition models

There is another criterion for classifying competition types, widely used by most environmentalists. In this criterion, the classification is made according to the type of competing entities.

- Intra-specific if it occurs between individuals of the same species.
- Inter-specific if it occurs between individuals of different species.

Competition models

The most obvious (and most investigated) case of **interspecific competition** is one in which populations of two species (say A and B) compete.

The effect of A on B is rarely equal to the effect of B on A.

At one extreme it will be the same, a situation of perfect reciprocity, but at the other extreme the effect of A on B is so dominant that the consequences of B presence for population A are negligible, a situation of **asymmetric competition**.

In nature, however, a population is affected by many populations (of different species) sharing the same resources. The term **diffuse competition** was introduced to designate the cumulative effect of these competitors on the population of interest. In this case it is assumed that no particular competitor.

Competition models

Consider now two species x, y that compete with each other for the same food supply.

Instead of analyzing specific equations, we follow a different procedure:

we consider a very broad class of equations, about which we assume only some qualitative characteristics. In this way a large generality of equations is obtained.

Growth equations of two species are written as follows.

$$\begin{cases} x' = M(x,y)x \\ y' = N(x,y)y \end{cases}$$

where M and N are continuum functions on the nonnegative variables x, y.

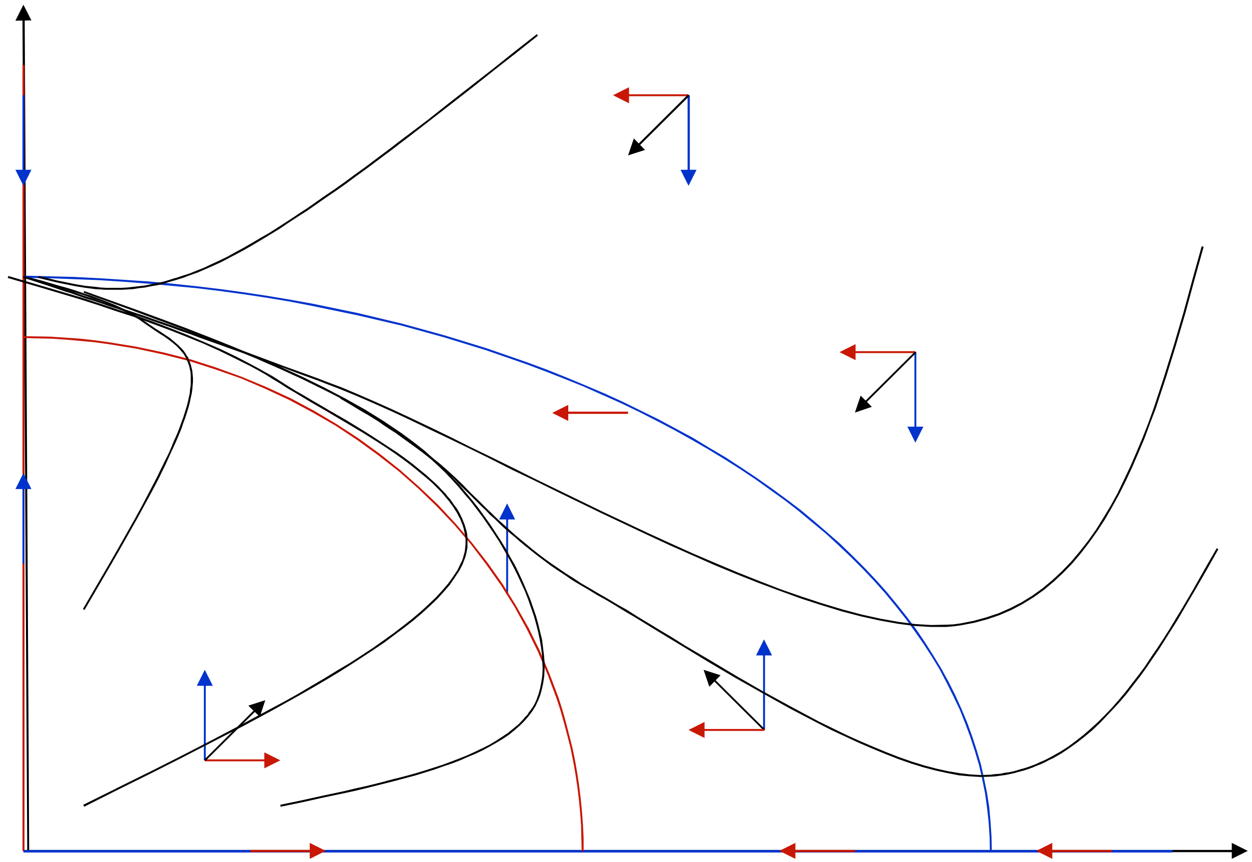
Competition models

We stablish the following assumptions:

- if one species increases the other decreases;
- if either population is very large, none of the species can increase
- in the absence of one of the species, the other has a positive growth rate to a certain population level and a negative growth rate thereafter.

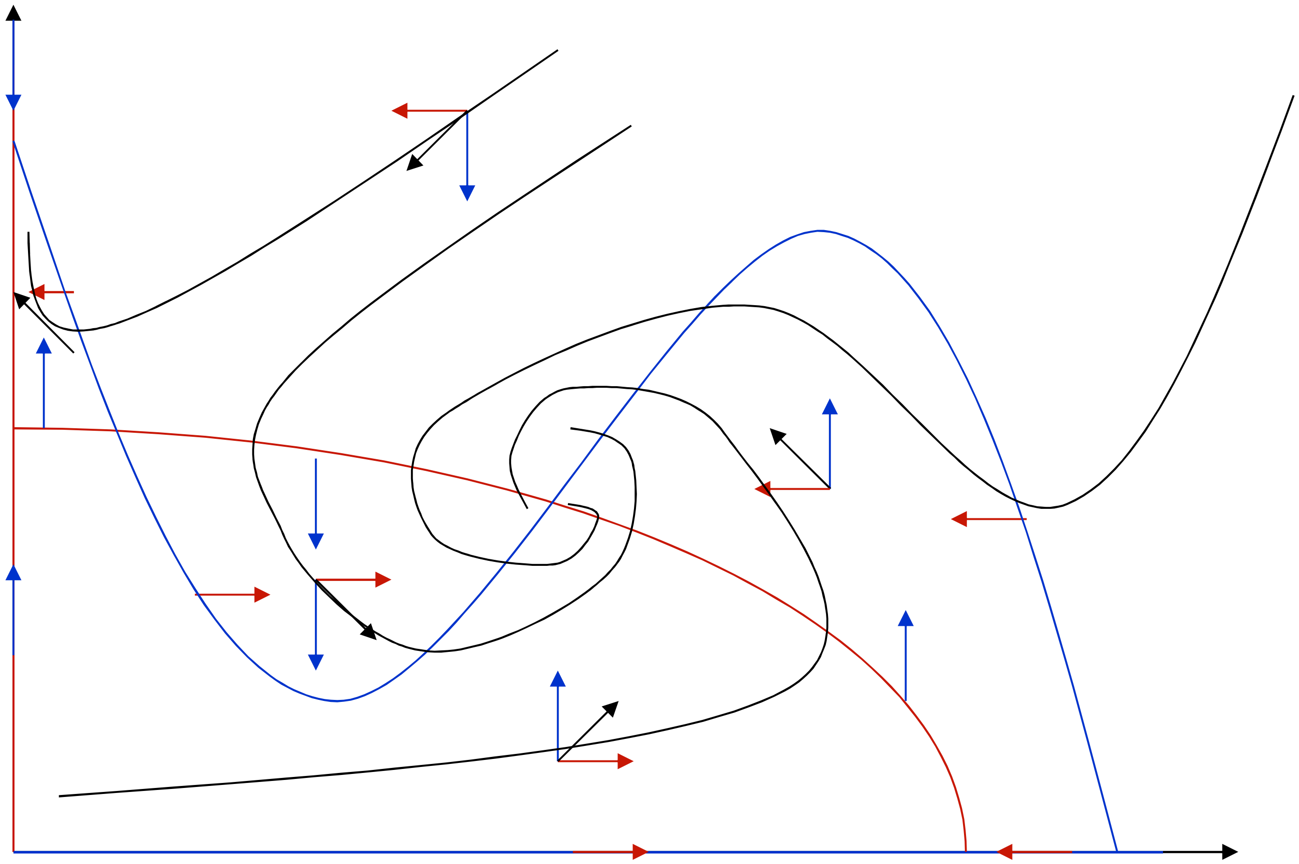
Competition models

The lines $M(x,y)$ and $N(x,y)$ do not intersect.



Competition models

The lines $M(x,y)$ and $N(x,y)$ intersect



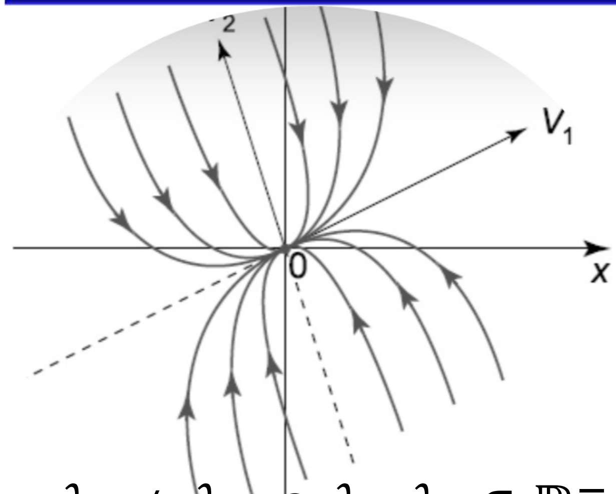
Competition models

Let

$$A(x,y)=\begin{bmatrix}\frac{\partial M}{\partial x}(x,y) & \frac{\partial M}{\partial y}(x,y)\\ \frac{\partial N}{\partial x}(x,y) & \frac{\partial N}{\partial y}(x,y)\end{bmatrix}$$

and λ_1 and λ_2 the eigenvalues of matrix $A(x^*,y^*)$ where (x^*,y^*) is an equilibrium point.

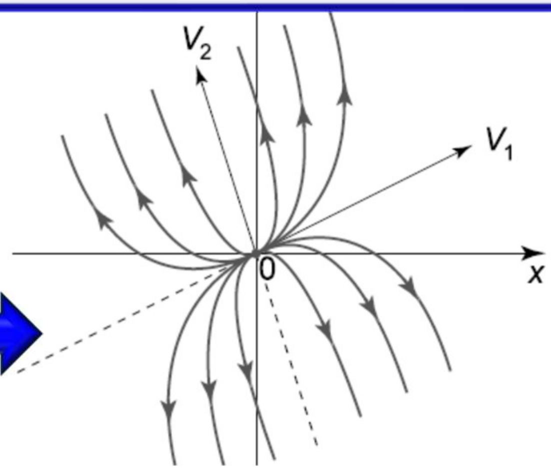
Competition models



Node

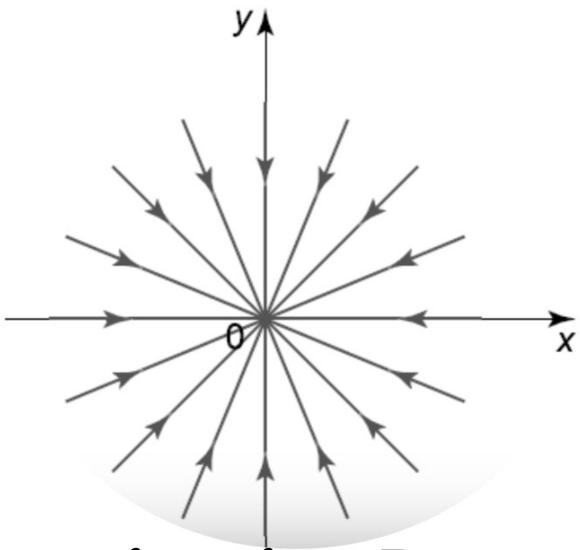
Stable

Unstable



$\lambda_1 \neq \lambda_2$ e $\lambda_1, \lambda_2 \in \mathbb{R}^-$

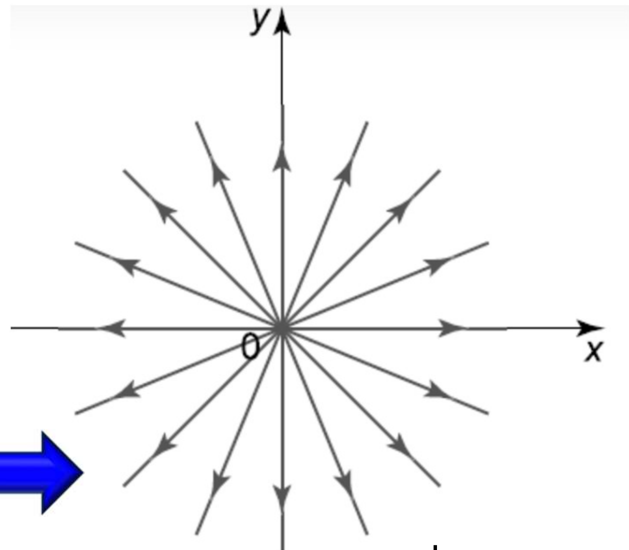
$\lambda_1 \neq \lambda_2$ e $\lambda_1, \lambda_2 \in \mathbb{R}^+$



Dicritical Node

Stable

Unstable

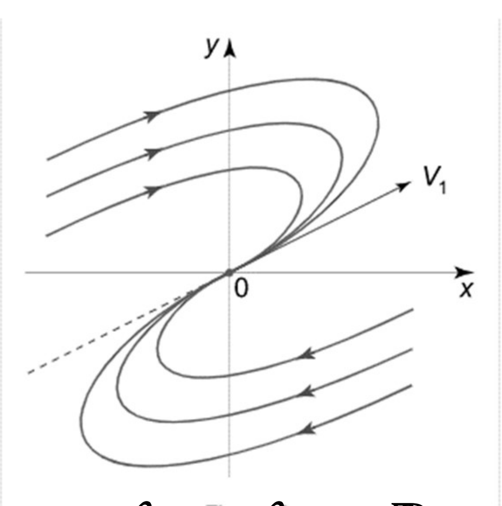


$\lambda_1 = \lambda_2 \in \mathbb{R}^-$

$\lambda_1 = \lambda_2 \in \mathbb{R}^+$

Competition models

Singular Node

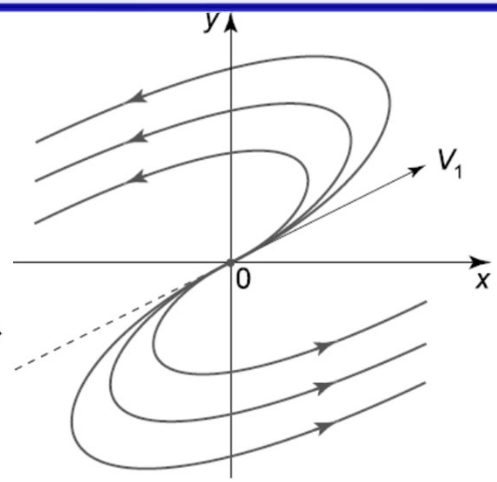


$\lambda_1 = \lambda_2 \in \mathbb{R}^-$

matrix no diagonalizable

Stable

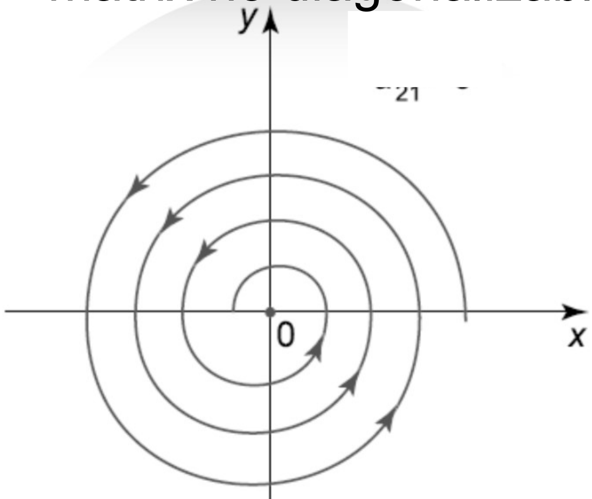
Unstable



$\lambda_1 = \lambda_2 \in \mathbb{R}^+$

matrix no diagonalizable

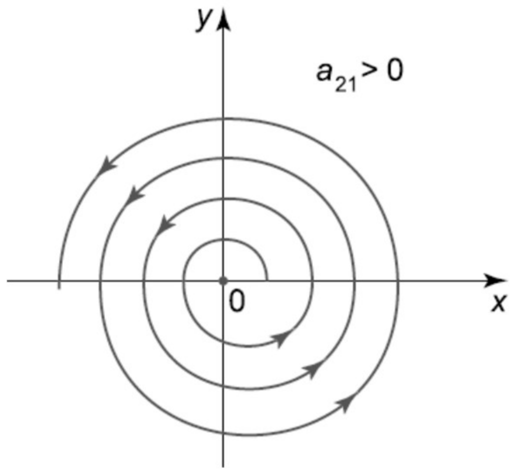
Focus



$\lambda = u \pm iv, u \in \mathbb{R}^-$

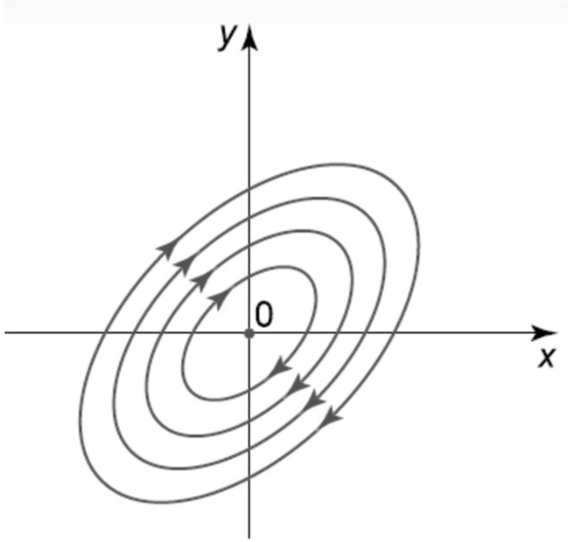
Stable

Unstable



$\lambda = u \pm iv, u \in \mathbb{R}^+$

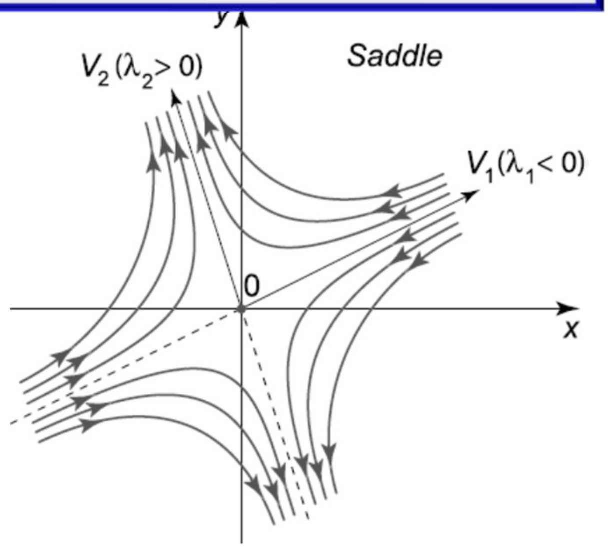
Competition models



$\lambda = \pm iv$

Center

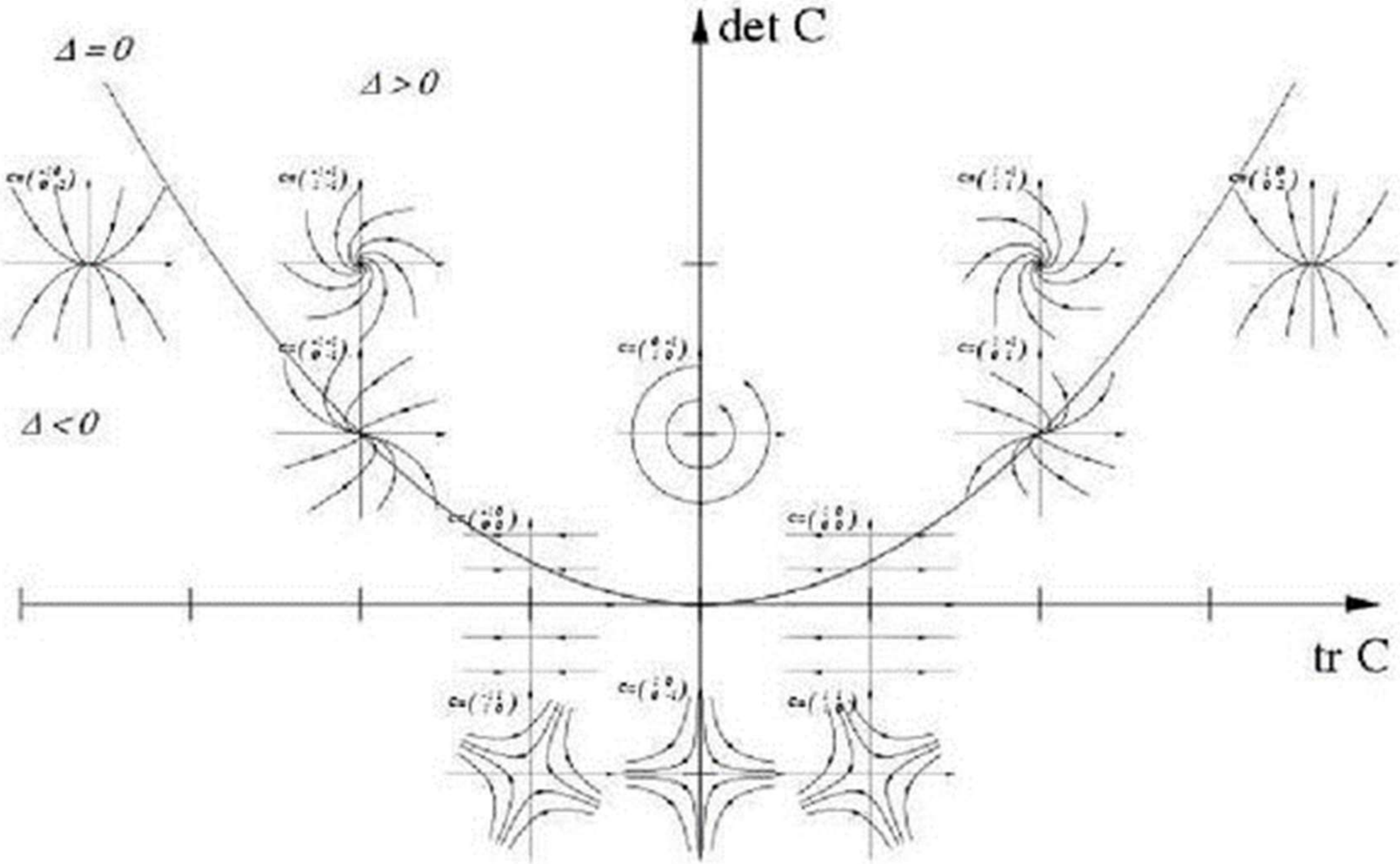
Saddle



$\lambda_1 \times \lambda_2 \in \mathbb{R}^-$

Competition models

$$\lambda^2 - trC\lambda + detC = 0$$



Competition models

Conclusion

The populations of two competing species always approach one of a finite number of possible limit populations.

Examining the equilibrium points for stability, we find the following result:

- A vertex in which curves have opposite slopes is asymptotically stable;
- The other equilibrium points that are asymptotically stable are $(\mathbf{0}, \boldsymbol{b})$ and $(\boldsymbol{a}, \mathbf{0})$;
- All other points are unstable,
- There must be at least one asymptotically stable equilibrium point.
- Any trajectory approaches a balance.

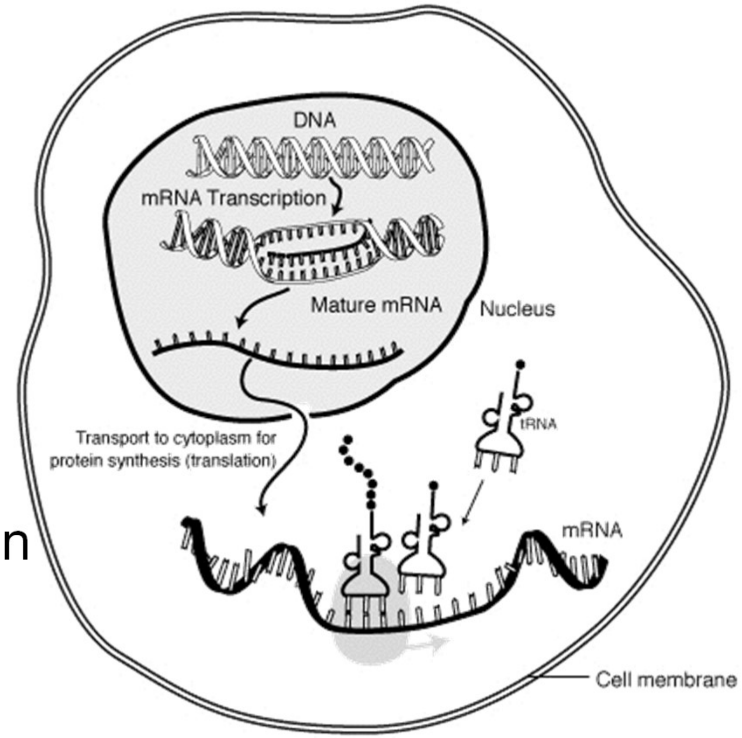
A mRNA and BMAL1 model

Let us to “travel” to the nucleus of a cell and analyze 2 important proteins:

mRNA and BMAL1

Consider the system

$$\begin{cases} x' = rx(1-x) - (1 - e^{-a}\ )y \\ y' = y[(1 - e^{-ax}) - D] \end{cases}$$



where:

- x represents the population of mRNA protein

  $(x(0) \geq 0)$

- y represents the population of BMAL1 protein

  $(y(0) \geq 0)$

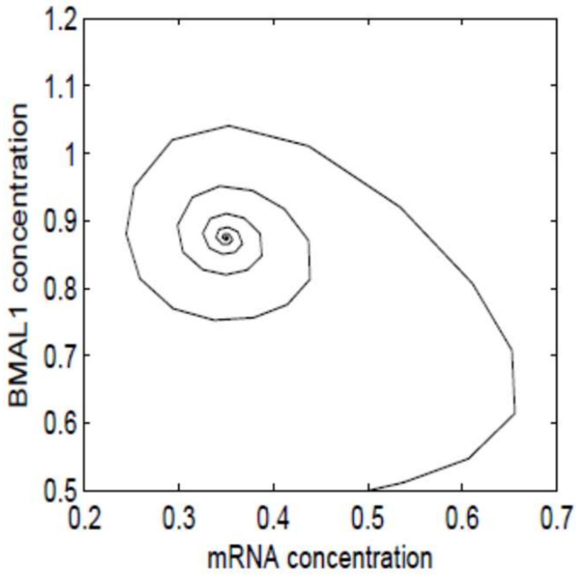
A mRNA and BMAL1 model

The equilibrium points are

- $E_0(0;0)$, always exist
- $E_A(1;0)$, always exist
- $E(x^*,y^*)$, where $x^* = \frac{1}{a}\ln\left(\frac{1}{1-D}\right)$ and $y^* = \frac{r}{D}x^*(1-x^*)$

Exist if $x^* < 1$ and $D < 1 - e^{-a}$

Stable solution for parameter values
$r = 2.5,\ a = 3.0,\ D = 0.65$



A mRNA and BMAL1 model

The linearization of the system

$$\begin{cases} x' = rx(1-x) - (1-e^{-ax})y \\ y' = y[(1-e^{-ax}) - D] \end{cases}$$

is given by

$$\begin{cases} u' = r\left[1 - 2x^* - \dfrac{a}{D}x^*(1-x^*)\right]u - Dv \\ v' = a(1-D)y^*u \end{cases}$$

The characteristic equation is given by $\lambda^2 - (TrA)\lambda + detA = 0$

and system will be stable if $Re(\lambda) < 0$

$$\Rightarrow \quad \boxed{1 - 2x^* - \frac{a}{D}x^*(1-x^*) < 0}$$

Tumor growth cancer model

Consider the system

$$\begin{cases} x' = 1 + a_1 x(1-x) - k_1 xy - k_2 x \\ y' = a_2 yz - a_3 y - k_3 xy \\ z' = a_4 z(1-z) - a_5 yz - a_6 z - k_4 xz \end{cases}$$

density of tumor cells

density of hunting predator cells

density of resulting cells

- a_1 is the growth rate of tumor cells,
- a_2 represents the conversion rate of the resulting cells to hunting predator cells,
- a_3 is the specific loss rates of hunting predator cells,
- a_4 represents the growth rate of resting cells,
- a_5 is the conversion rate of resting cells to hunting predator cells,
- a_6 is the specific loss rates of the resting cells,
- k_1 is the rate of killing of tumor cells by hunting cells,
- k_2 is the specific loss rates of tumor cells,
- k_3 represents the rate of killing of hunting predator cells by tumor cells,
- k_4 represents rate of killing of resting cells by tumor cells.

Tumor growth cancer model

The equilibrium points of the system

$$\begin{cases} x' = 1 + a_1 x(1-x) - k_1 xy - k_2 x \\ y' = a_2 yz - a_3 y - k_3 xy \\ z' = a_4 z(1-z) - a_5 yz - a_6 z - k_4 xz \end{cases}$$

are:

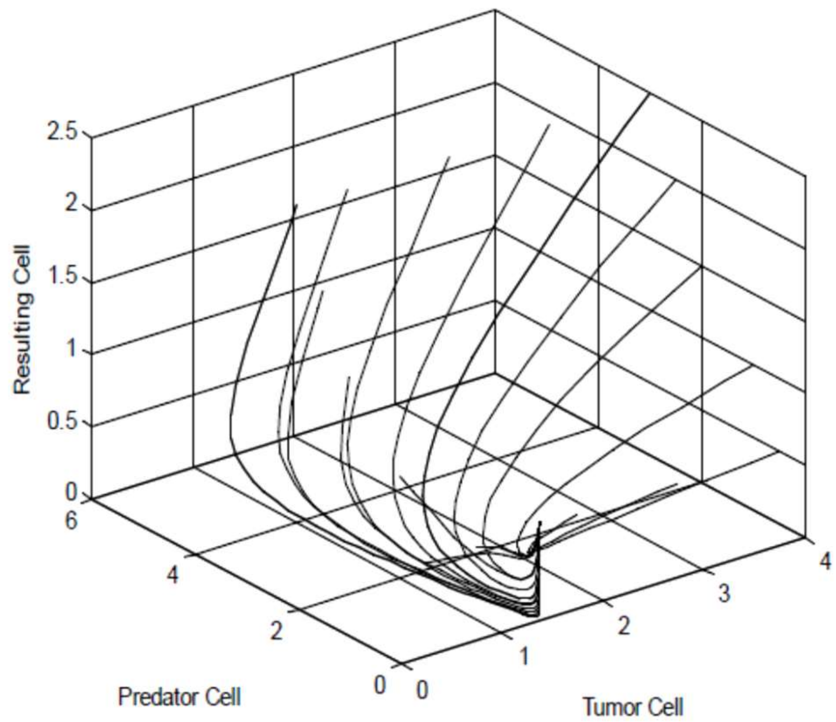
- $E_0(0,0,0)$
- $E_1(x_1,0,0)$ where $x_1 = \frac{1}{2}\left[\left(1-\frac{k_2}{a_1}\right) + \sqrt{\left(1-\frac{k_2}{a_1}\right)^2 + \frac{4}{a_1}}\right]$ $\quad (a_1 > k_2)$
- $E_2(x_2,0,z_2)$ where $x_2 = \frac{1}{2}\left[\left(1-\frac{k_2}{a_1}\right) + \sqrt{\left(1-\frac{k_2}{a_1}\right)^2 + \frac{4}{a_1}}\right]$ and

  $z_2 = 1 - \frac{a_6}{a_4} - \frac{k_4}{a_4}x_2$ $\quad (a_1 > k_2$ and $a_4 > a_6 + k_4 x_2)$
- $E_3(x_3,y_3,z_3)$ where $y_3 = \frac{1+a_1x_3(1-x_3)-k_2x_3}{k_1x_3}$ and $z_3 = \frac{a_3+k_3x_3}{a_2}$ $\quad (a_1 > k_2)$

Tumor growth cancer model

The equilibrium point $E_3(x_3, y_3, z_3)$ is globaly asymptotically stable

Example: $E_3(1.3213, 0.5656, 0.1186)$



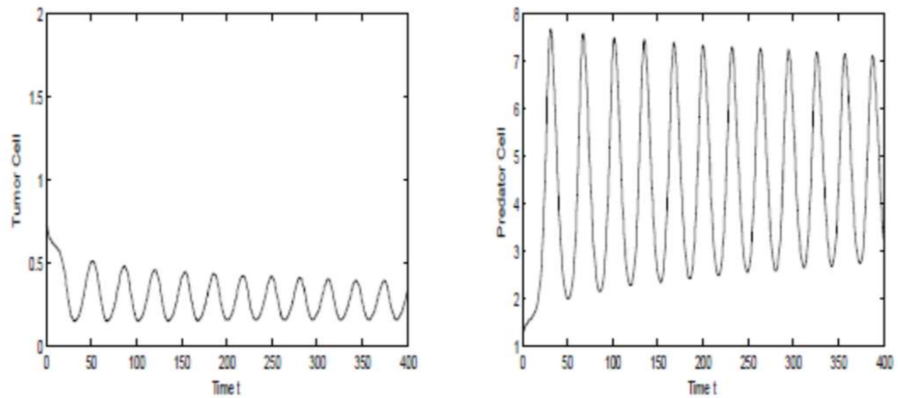
Tumor growth cancer model

The equilibrium point $E_3(x_3, y_3, z_3)$ is locally asymptotically stable if

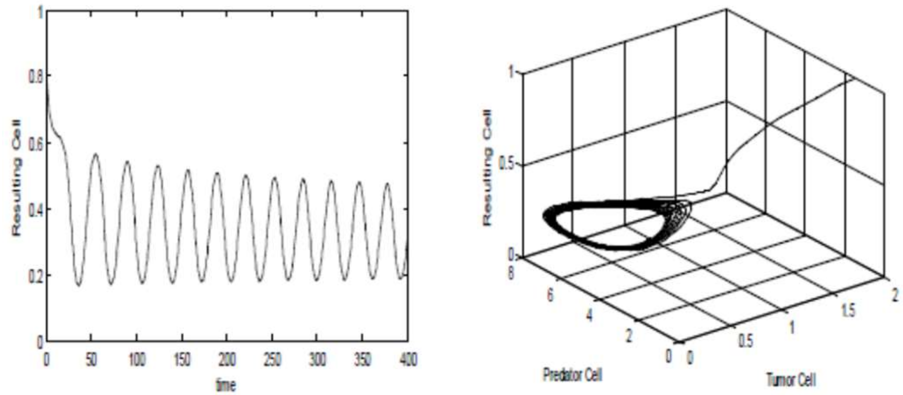
- $a_2 a_5(1 + a_1 x_3^2) > k_1 x_3^2(a_4 k_3 + a_2 k_4)$
- $a_2 k_4 x_3 z_3 > (1 + a_1 x_3^2)k_3$

Example:

- $a_1 = 0{,}6 = a_4$
- $a_2 = 0{,}99$
- $a_3 = 0{,}1$
- $a_5 = 0{,}06$
- $a_6 = 0{,}118$
- $k_1 = 0{,}9$
- $k_2 = 0{,}5$
- $k_3 = 0{,}854$
- $k_4 = 0{,}02$



E_3(1.3213, 0.5656, 0.1186)



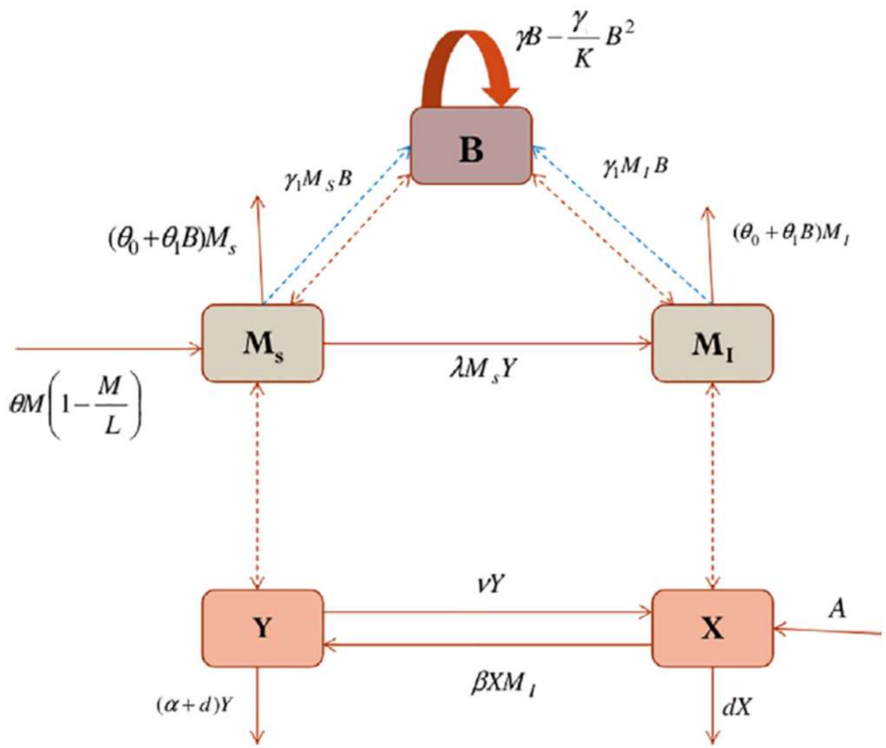
Biolarvicide vs malaria model

Consider the system

$$\begin{cases} x' = a - \beta x m_I - dx + vy \\ y' = \beta x m_I - (v - \alpha - d)y \\ m'_S = \theta M\left(1 - \dfrac{m_S + m_I}{L}\right) - (\theta_0 + \theta_1 B)m_S - \lambda m_s y \\ m'_I = \lambda m_I y - (\theta_0 + \theta_1 B)m_I \\ B' = \gamma B\left(1 - \dfrac{B}{k}\right) + \gamma_1(m_S + m_I)B \end{cases}$$

- x represent the susceptible humans
- y represent the infected humans
- m_S represent the susceptible mosquitoes
- m_I represent the infected mosquitoes
- B represent the biolarvicide population

Biolarvicide vs malaria model



The direction of each solid line represents movement of population along that line within the same species.

Example: $\beta x m_I$ is a removal from x population and an addition to y population.

The bi-directional dotted lines between boxes indicates a mass-action interaction.

The single directional dotted line indicates increase of bacteria population.

Biolarvicide vs malaria model

The equilibrium points of the system

- $E_0\left(\frac{a}{d},0,0,0,0\right)$ ⟵ Disease free
  Unstable
- $E_1\left(\frac{a}{d},0,0,\frac{L(\theta-\theta_0)}{\theta},0\right)$ ⟵ Disease free
  Unstable
- $E_2\left(\frac{a}{d},0,0,0,K\right)$ ⟵ Disease free
  Unstable if $\frac{(\theta-\theta_0)}{\theta_1} > k$
- $E_3\left(x^*,y^*,m_S^*,m_I^*,0\right)$ ⟵ Endemic
  Unstable
- $E_4\left(\frac{a}{d},0,m_S^*,0,B^*\right)$ ⟵ Disease free
  Stable under conditions
- $E_5\left(x^*,y^*,m_S^*,m_I^*,B^*\right)$ ⟵ Endemic
  Stable under conditions

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