A LINEAR DIFFERENCE EQUATION OF MIXED TYPE

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INTRODUCTION

The aim of this lesson is to study oscillatory and asymptotic behavior for a scalar linear difference equation of mixed type

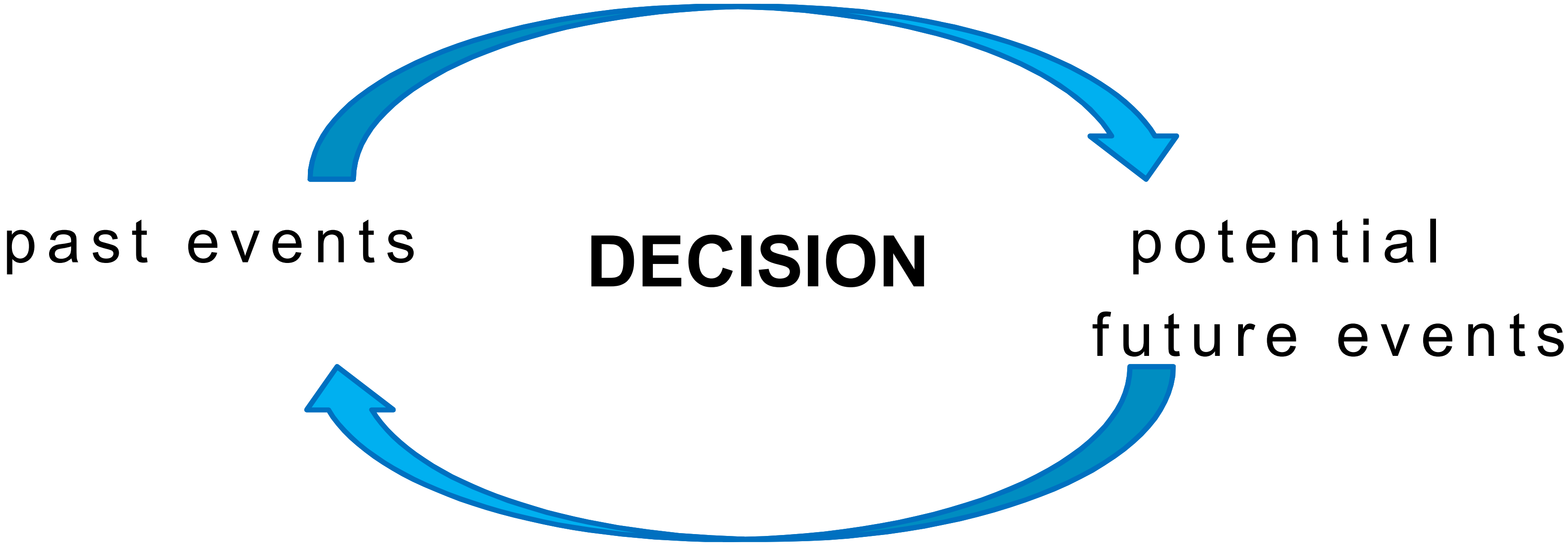
$$\Delta x(n) + \sum_{k=-p}^{q} a_k(n)x(n+k) = 0, \quad n > n_0,$$

- $\Delta x(n) = x(n+1) - x(n)$ is the difference operator
- $\{a_k(n)\}$ are sequences of real numbers
- $p > 0,\ q \geqslant 0$

INTRODUCTION

Differential equations with delayed and advanced arguments occur in many problems:

- Economy
- Biology
- Physics



INTRODUCTION

Why is this kind of equations a challenge?

It is well known that the solutions of these types of equations cannot be obtained in closed-form. It is not quite clear how to formulate an initial value problem for such equations and the existence and uniqueness of solutions becomes a complicated issue. To study the oscillation of solutions of differential equations, we need to assume that there exists a solution of such equations on the half line.

INTRODUCTION

The characteristic equation of the equation

$$\Delta x(t) + \sum_{k=-p}^{q} a_k x(n+k) = 0, \qquad n > n_0$$

is

$$\lambda - 1 + \sum_{k=-p}^{q} a_k \lambda^{kt} = 0.$$

The difference equation is oscillatory if and only if the characteristic equation has no positive roots.

OSCILLATORY BEHAVIOR

Theorem 1:

- $a_k(n)$ are nonnegative for all $k \in \{-p, \dots, q\}$ and $n > n_0$.
- $\sum_{k=-p}^{-1} a_k(n+1) + \sum_{k=-p}^{0} a_k(n) \geqslant 1$

Then all solutions of the difference equation

$$\Delta x(n) + \sum_{k=-p}^{q} a_k(n) x(n+k) = 0, \quad n > n_0,$$

are oscillatory.

OSCILLATORY BEHAVIOR

Draft of the proof:

$\{x(n)\}$ is eventually positive $\longrightarrow$ $\{x(n)\}$ is decreasing

$$0 = x(n+2) - x(n+1) + \sum_{k=-p}^{q} a_k(n+1)x(n+k+1)$$

$$> \; -x(n+1) + \sum_{k=-p}^{-1} a_k(n+1)x(n+k+1)$$

$$\geqslant \; -x(n+1) + \sum_{k=-p}^{-1} a_k(n+1)x(n)$$

$$x(n+1) > \sum_{k=-p}^{-1} a_k(n+1)x(n)$$

OSCILLATORY BEHAVIOR

Draft of the proof (cont.):

On the other hand

$$0 = x(n+1) - x(n) + \sum_{k=-p}^{q} a_k(n)x(n+k)$$

$$\geqslant x(n+1) - x(n) + \sum_{k=-p}^{0} a_k(n)x(n+k)$$

$$\geqslant x(n+1) - x(n) + \sum_{k=-p}^{0} a_k(n)x(n)$$

$$x(n+1) \leqslant x(n) - \sum_{k=-p}^{0} a_k(n)x(n)$$

OSCILLATORY BEHAVIOR

Draft of the proof (cont.):

$$\begin{cases} x(n+1) > \displaystyle\sum_{k=-p}^{-1} a_k(n+1)x(n) \\ x(n+1) \leqslant x(n) - \displaystyle\sum_{k=-p}^{0} a_k(n)x(n) \end{cases}$$

$$\sum_{k=-p}^{-1} a_k(n+1)x(n) < x(n) - \sum_{k=-p}^{0} a_k(n)x(n)$$

$$\sum_{k=-p}^{-1} a_k(n+1) + \sum_{k=-p}^{0} a_k(n) < 1$$

Contradition

OSCILLATORY BEHAVIOR

Corollary 1:

- $a_k(n)$ are nonnegative for all $k \in \{-p, \dots, q\}$ and $n > n_0$.

- There exist a $m \in \{-p, \dots, 0\}$ such that

$$\sum_{k=-p}^{m} a_k(n) > 1$$

Then all solutions of the difference equation

$$\Delta x(n) + \sum_{k=-p}^{q} a_k(n) x(n+k) = 0, \quad n > n_0,$$

are oscillatory.

OSCILLATORY BEHAVIOR

Corollary 2:

- $a_k(n)$ are nonnegative for all $k \in \{-p, \ldots, q\}$ and $n > n_0$.

- $a_0(n) \geqslant 1 - p \min\{a_k(n) \colon\ k = -p, \ldots, -1\} > 0$

Then all solutions of the difference equation

$$\Delta x(n) + \sum_{k=-p}^{q} a_k(n) x(n+k) = 0, \quad n > n_0,$$

are oscillatory.

OSCILLATORY BEHAVIOR

Corollary 3:

- $a_k(n)$ are nonnegative for all $k \in \{-p, \ldots, q\}$ and $n > n_0$.
- $a_{-p}(n) < \ldots < a_{-1}(n) \quad \text{and} \quad a_0(n) > 1 - pa_{-p}(n)$

Then all solutions of the difference equation

$$\Delta x(n) + \sum_{k=-p}^{q} a_k(n) x(n+k) = 0, \qquad n > n_0,$$

are oscillatory.

OSCILLATORY BEHAVIOR

Corollary 4:

- $a_k(n)$ are nonnegative for all $k \in \{-p, \ldots, q\}$ and $n > n_0$.
- $a_{-p}(n) > \ldots > a_{-1}(n) \quad \text{and} \quad a_0(n) > 1 - pa_{-1}(n)$

Then all solutions of the difference equation

$$\Delta x(n) + \sum_{k=-p}^{q} a_k(n) x(n+k) = 0, \quad n > n_0,$$

are oscillatory.

OSCILLATORY BEHAVIOR

Example:

$$\Delta x(n) + \sum_{k=-2}^{0} \frac{n-1}{|k-2|(n+1)} x(n+k) + \sum_{k=1}^{q} b_k(n) x(n+k) = 0$$

where:

- $n \geqslant 4$
- $b_k(n)$ is a nonnegative sequence
- $q \geqslant 1$

Since

$$\frac{n}{4(n+2)} + \frac{n}{3(n+2)} + \frac{n-1}{4(n+1)} + \frac{n-1}{3(n+1)} + \frac{n-1}{2(n+1)} > 1.$$

for $n > 2$, by the **Theorem 1**, we can conclude that equation is oscillatory.

OSCILLATORY BEHAVIOR

Theorem 2:

- $a_k(n)$ are nonpositive for all $k \in \{-p, \ldots, q\}$, $q \geq 2$ and $n > n_0$.
- $\sum_{k=1}^{q} a_k(n) + \sum_{k=2}^{q} a_k(n-1) \leqslant -1$

Then all solutions of the difference equation

$$\Delta x(n) + \sum_{k=-p}^{q} a_k(n) x(n+k) = 0, \quad n > n_0,$$

are oscillatory.

OSCILLATORY BEHAVIOR

Corollary 1:

- $a_k(n)$ are nonpositive for all $k \in \{-p, \ldots, q\}$ and $n > n_0$.

- There exist a $m \in \{1, \ldots, q\}$ such that

$$\sum_{k=1}^{m} a_k(n) \leqslant -1$$

Then all solutions of the difference equation

$$\Delta x(n) + \sum_{k=-p}^{q} a_k(n) x(n+k) = 0, \quad n > n_0,$$

are oscillatory.

OSCILLATORY BEHAVIOR

Corollary 2:

- ❑ $a_k(n)$ are nonpositive for all $k \in \{-p, \ldots, q\}$ and $n > n_0$.

- ❑ $a_1(n) \leqslant -1 - q \max\{a_k(n) \colon \; k = 2, \ldots, q\}$

Then all solutions of the difference equation

$$\Delta x(n) + \sum_{k=-p}^{q} a_k(n) x(n+k) = 0, \qquad n > n_0,$$

are oscillatory.

OSCILLATORY BEHAVIOR

Corollary 3:

- $a_k(n)$ are nonpositive for all $k \in \{-p, \ldots, q\}$ and $n > n_0$.
- $a_2(n) < \ldots < a_q(n) \quad \text{and} \quad a_1(n) < -1 - q a_q(n)$

Then all solutions of the difference equation

$$\Delta x(n) + \sum_{k=-p}^{q} a_k(n) x(n+k) = 0, \qquad n > n_0,$$

are oscillatory.

OSCILLATORY BEHAVIOR

Corollary 4:

- $a_k(n)$ are nonpositive for all $k \in \{-p, \ldots, q\}$ and $n > n_0$.
- $a_2(n) > \ldots > a_q(n) \quad \text{and} \quad a_1(n) < -1 - q a_2(n)$

Then all solutions of the difference equation

$$\Delta x(n) + \sum_{k=-p}^{q} a_k(n) x(n+k) = 0, \quad n > n_0,$$

are oscillatory.

OSCILLATORY BEHAVIOR

Example:

$$\Delta x(n) + \sum_{k=-p}^{0} c_k(n)x(n+k) + \sum_{k=1}^{3} \frac{k(e^{-n}-1)}{k+1} x(n+k) = 0$$

where:

- $n \geqslant 1$
- $c_k(n)$ is a nonpositive sequence

Since

$$\frac{e^{-n}-1}{2} + \frac{2(e^{-n}-1)}{3} + \frac{3(e^{-n}-1)}{4} + \frac{2(e^{-(n-1)}-1)}{3} + \frac{3(e^{-(n-1)}-1)}{4} < -1$$

by the **Theorem 2**, we can conclude that equation is oscillatory.

OSCILLATORY BEHAVIOR

Theorem 3:

- For each $k \in \{-p, \dots, q\}$ there exist the limit

$$\lim_{n\to\infty} a_k(n) = a_k \neq 0$$

- All roots $\lambda_1, \lambda_2, \dots, \lambda_{q+p}$ of the equation

$$\lambda - 1 + \sum_{k=-p}^{q} a_k \lambda^k = 0$$

satisfy $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_{q+p}|$ and n of them are negative

Then all solutions of the difference equation

$$\Delta x(n) + \sum_{k=-p}^{q} a_k(n) x(n+k) = 0, \quad n > n_0,$$

has n oscillatory solutions.

OSCILLATORY BEHAVIOR

Draft of the proof:

λ_k a real negative root of $\lambda - 1 + \sum_{k=-p}^{q} a_k \lambda^k = 0$

Perron Theorem

$$\Delta x(n) + \sum_{k=-p}^{q} a_k(n) x(n+k) = 0, \quad n > n_0,$$

has a solution such that $\lim_{n\to\infty} \frac{u_k(n+1)}{u_k(n)} = \lambda_k < 0$

the solution is necessarily an oscillatory solution

OSCILLATORY BEHAVIOR

Example:

$$\Delta x(n) - \frac{6n}{n+6}x(n-2) + \frac{5n+1}{n}x(n-1) + 6(\mathrm{e}^{-n}+1)x(n)$$
$$+ 6(\mathrm{e}^{-n}-1)x(n+1) + \frac{n}{n+1}x(n+2) = 0,$$

Since:

$$a_{-2}(n) = -\frac{6n}{n+6} \xrightarrow[n\to\infty]{} -6 \qquad a_{-1}(n) = \frac{5n+1}{n} \xrightarrow[n\to\infty]{} 5$$

$$a_0(n) = 6(\mathrm{e}^{-n}+1) \xrightarrow[n\to\infty]{} 6 \qquad a_1(n) = 6(\mathrm{e}^{-n}-1) \xrightarrow[n\to\infty]{} -6$$

$$a_2(n) = \frac{n}{n+1} \xrightarrow[n\to\infty]{} 1$$

$$\lambda - 1 - 6\lambda^{-2} + 5\lambda^{-1} + 6 - 6\lambda + \lambda^2 = 0$$

OSCILLATORY BEHAVIOR

Example (cont.):

$$\lambda - 1 - 6\lambda^{-2} + 5\lambda^{-1} + 6 - 6\lambda + \lambda^2 = 0$$

$$\lambda_1 = 3 \qquad \lambda_2 = 2 \qquad \lambda_3 = 1 \qquad \lambda_4 = -1$$

Theorem 3

$$\Delta x(n) - \frac{6n}{n+6}x(n-2) + \frac{5n+1}{n}x(n-1) + 6(\mathrm{e}^{-n}+1)x(n)$$
$$+ 6(\mathrm{e}^{-n}-1)x(n+1) + \frac{n}{n+1}x(n+2) = 0,$$

has an oscillatory solution.

NONOSCILLATORY BEHAVIOR

On this section, we will study the equation

$$\Delta x(n) + \sum_{k=-p}^{q} a_k x(n+k) = 0, \quad n \geqslant 1.$$

According to Krisztin in ***Nonoscillation for functional differential equations of mixed type. J. Math. Anal. Appl. 245 (2000), 326–345***, the equation

$$\Delta x(n) + \sum_{k=-p}^{q} a_k x(n+k) = 0$$

is nonoscillatory if there exists $\lambda \in \mathbb{R}^+$ such that

$$\lambda - 1 + \sum_{k=-p}^{q} a_k \lambda^k = 0$$

NONOSCILLATORY BEHAVIOR

Theorem 4:

- $a_{-p}a_q < 0$

Then

$$\Delta x(n) + \sum_{k=-p}^{q} a_k x(n+k) = 0$$

is nonoscillatory.

NONOSCILLATORY BEHAVIOR

Draft of the proof:

$$\begin{cases} a_{-p} < 0 \\ a_q > 0. \end{cases}$$

$$\left.\begin{aligned} N(\lambda) &= 1 - \lambda - \sum_{k=-p}^{-1} a_k \lambda^k - \sum_{k=0}^{q} a_k \lambda^k \xrightarrow[\lambda \to 0^+]{} \infty \\ N(\lambda) &= 1 - \lambda - \sum_{k=-p}^{-1} a_k \lambda^k - \sum_{k=0}^{q} a_k \lambda^k \xrightarrow[\lambda \to \infty]{} -\infty \end{aligned}\right\}$$

By the continuity, there exist λ_0 such that $N(\lambda_0) = 0$

NONOSCILLATORY BEHAVIOR

Example:

$$\Delta x(n) + ax(n-1) + (1-3a)x(n) - (1-3a)x(n+1) - ax(n+2) = 0, \quad n \geqslant 1$$

Since

$$a_{-1} = a = -a_2$$

Then the equation has a nonoscillatory solution.

$$x(n) = n$$

is a nonoscillatory solution

NONOSCILLATORY BEHAVIOR

Theorem 5:

- $a_k > 0$
- $$\max\left\{\left(p\sum_{k=-p}^{q} a_k\right)^{1/(p+1)}, \left(\sum_{k=-p}^{q} a_k\right)^{1/(p+1)} \frac{p+1}{p^{p/(p+1)}}\right\} < 1$$

Then

$$\Delta x(n) + \sum_{k=-p}^{q} a_k x(n+k) = 0$$

is nonoscillatory.

NONOSCILLATORY BEHAVIOR

Draft of the proof:

$$N(\lambda) = 1 - \lambda - \sum_{k=-p}^{q} a_k \lambda^k < 1 - \lambda < 0 \text{ for } \lambda \geqslant 1.$$

$$\lambda < 1$$

$$N(\lambda) = 1 - \lambda - \sum_{k=-p}^{q} a_k \lambda^k > \underbrace{1 - \lambda - \lambda^{-p} \sum_{k=-p}^{q} a_k}_{\text{has a maximum}}$$

$$\lambda_0 = \left(p \sum_{k=-p}^{q} a_k\right)^{1/(p+1)}$$

$$f(\lambda_0) = 1 - \left(\sum_{k=-p}^{q} a_k\right)^{1/(p+1)} \frac{p+1}{p^{p/(p+1)}} > 0 \Longrightarrow N(\lambda_0) > 0$$

NONOSCILLATORY BEHAVIOR

Theorem 6:

- $a_k < 0$
- $$\min\left\{\left(-q\sum_{k=-p}^{q} a_k\right)^{-1/(q-1)}, \left(-\sum_{k=-p}^{q} a_k\right)^{-1/(q-1)} \frac{q+1}{q^{q/(q-1)}}\right\} \geqslant 1$$

Then

$$\Delta x(n) + \sum_{k=-p}^{q} a_k x(n+k) = 0$$

is nonoscillatory.

NONOSCILLATORY BEHAVIOR

Draft of the proof:

$$N(\lambda) = 1 - \lambda - \sum_{k=-p}^{q} a_k \lambda^k > 1 - \lambda > 0 \text{ for } \lambda \leqslant 1.$$

$$\lambda > 1$$

$$N(\lambda) = 1 - \lambda - \sum_{k=-p}^{q} a_k \lambda^k > \underbrace{1 - \lambda - \lambda^q \sum_{k=-p}^{q} a_k}_{\text{has a minimum}}$$

$$\lambda_1 = \left(-q \sum_{k=-p}^{q} a_k\right)^{-1/(q-1)}$$

$$g(\lambda_1) = 1 - \left(-\sum_{k=-p}^{q} a_k\right)^{-1/(q-1)} \frac{q+1}{q^{q/(q-1)}} \leqslant 0 \Longrightarrow N(\lambda_1) > 0$$

NONOSCILLATORY BEHAVIOR

Example: $p = 1$

$$\Delta x(n) + \frac{253}{2048}x(n-1) + \frac{1}{512}x(n) + \frac{1}{256}x(n+1) + \frac{1}{64}x(n+3) = 0$$

Since

$$\left(p\sum_{k=-p}^{q} a_k\right)^{1/(p+1)} = \sqrt{\frac{297}{2048}} < 1$$

$$\left(\sum_{k=-p}^{q} a_k\right)^{1/(p+1)} \frac{p+1}{p^{p/(p+1)}} = \sqrt{\frac{297}{512}} < 1$$

$x(n) = 2^{-n}$

is a nonoscillatory solution

Then the equation has a nonoscillatory solution.

ASYMPTOTIC BEHAVIOR

Theorem 7:

- $a_k(n) \geqslant 0$
- $\sum_{n=n_1}^{\infty} \sum_{k=-p}^{q} a_k(n) = \infty$

If x is an eventually positive solution of

$$\Delta x(n) + \sum_{k=-p}^{q} a_k(n) x(n+k) = 0, \quad n > n_0,$$

then

$$\lim_{n \to \infty} x(n) = 0$$

ASYMPTOTIC BEHAVIOR

Draft of the proof:

$x(n) > 0 \longrightarrow \Delta x(n) < 0 \longrightarrow x(n)$ is decreasing

$$\lim_{n\to\infty} x(n) = d > 0$$

$$x(n) > d$$

$$\Delta x(n) = -\sum_{k=-p}^{q} a_k(n)x(n+k) < -d\sum_{k=-p}^{q} a_k(n)$$

$$x(n+1) < x(n_1) - d\sum_{i=n_1}^{n}\sum_{k=-p}^{q} a_k(i) \underset{n\to\infty}{\longrightarrow} -\infty$$

Contradition

ASYMPTOTIC BEHAVIOR

Theorem 8:

- $a_k(n) \leqslant 0$
- $\sum_{n=n_2}^{\infty} \sum_{k=-p}^{q} a_k(n) = -\infty$

If x is an eventually positive solution of

$$\Delta x(n) + \sum_{k=-p}^{q} a_k(n) x(n+k) = 0, \quad n > n_0,$$

then

$$\lim_{n\to\infty} x(n) = \infty$$

ASYMPTOTIC BEHAVIOR

Draft of the proof:

$x(n) > 0 \longrightarrow \Delta x(n) > 0 \longrightarrow x(n)$ is increasing

$$\Delta x(n) = -\sum_{k=-p}^{q} a_k(n)x(n+k) \geqslant -x(n-p)\sum_{k=-p}^{q} a_k(n)$$

$$x(n+1) > x(n-p)\left(1 - \sum_{i=n-p}^{n}\sum_{k=-p}^{q} a_k(i)\right) \xrightarrow[n\to\infty]{} \infty$$

ASYMPTOTIC BEHAVIOR

Example:

$k, l \geqslant 1$

$$\Delta x(n) + 3^{-1-(2nk-k^2)/(2n+1)} x(n-k) + 3^{-1+(2nl+l^2)/(2n+1)} x(n+l) = 0, \quad n \geqslant 1$$

Since

$$\sum_{n=1}^{\infty} \left(3^{-1-(2nk-k^2)/(2n+1)} + 3^{-1+(2nl+l^2)/(2n+1)}\right) = \infty$$

If the equation has a positive solution,

then

$$\lim_{n\to\infty} x(n) = 0$$

$$x(n) = 3^{-n^2/(2n+1)}$$

is a positive solution

ASYMPTOTIC BEHAVIOR

Let us to rewrite the equation as

$$\Delta x(n) + \sum_{k=1}^{p} a_k(n)\, x(n-k) + \sum_{k=1}^{q} b_k(n)\, x(n+k) = 0$$

where:

- $a_k(n)$ and $b_k(n)$ are real sequences
- $p, q > 0$

ASYMPTOTIC BEHAVIOR

Lemma:

- $x(n)$ is a positive solution
- $Q(n) = \sum_{k=1}^{p} a_k(n) + \sum_{k=1}^{q} b_k(n) \neq 1$

Then $x(n)$ satisfy the equation

$$x(n) = x(n_0) \prod_{j=n_0}^{n-1} (1 - Q(j)) + \left[\sum_{\ell=n_0}^{n} \prod_{j=\ell+1}^{n-1} (1 - Q(j)) \left(\sum_{k=1}^{p} a_k(\ell) \sum_{i=\ell-k}^{\ell-1} E_x(i) - \sum_{k=1}^{q} b_k(\ell) \sum_{i=\ell}^{\ell+k-1} E_x(i) \right) \right]$$

where

$$E_x(n) = \sum_{k=1}^{p} a_k(n)\, x(n-k) + \sum_{k=1}^{q} b_k(n)\, x(n+k)$$

ASYMPTOTIC BEHAVIOR

Theorem 9:

- $$\prod_{j=n_0}^{n-1}|1-Q(j)|+\sum_{\ell=n_0}^{n}\prod_{j=\ell+1}^{n-1}|1-Q(j)| \times\left[\sum_{k=1}^{p}|a_k(\ell)|\sum_{i=\ell-k}^{\ell-1}\left(\sum_{k=1}^{p}|a_k(i)|+\sum_{k=1}^{q}|b_k(i)|\right)\right] + \sum_{\ell=n_0}^{n}\prod_{j=\ell+1}^{n-1}|1-Q(j)| \left[\sum_{k=1}^{q}|b_k(\ell)|\sum_{i=\ell}^{\ell+k-1}\left(\sum_{k=1}^{p}|a_k(i)|+\sum_{k=1}^{q}|b_k(i)|\right)\right]=c<1$$
- $$\lim_{n\to+\infty}\prod_{j=n_0}^{n-1}|1-Q(j)|=0$$

Then, any solution converge to zero.

ASYMPTOTIC BEHAVIOR

Draft of the proof:

$$(Tx)(n) = x(n_0)\prod_{j=n_0}^{n-1}(1-Q(j)) + \sum_{\ell=n_0}^{n}\prod_{j=\ell+1}^{n-1}(1-Q(j))$$

$$\left(\sum_{k=1}^{p} a_k(\ell)\sum_{i=\ell-k}^{\ell-1}E_x(i) - \sum_{k=1}^{q} b_k(\ell)\sum_{i=\ell}^{\ell+k-1}E_x(i)\right)$$

is a contraction ⟶ T has a fixed point ⟶ $(Tx)(n) \to 0$

$$\lim_{n\to\infty} x(n) = 0$$

ASYMPTOTIC BEHAVIOR

Theorem 10:

- $$\prod_{j=n_0}^{n-1}|1-Q(j)|+\sum_{\ell=n_0}^{n}\prod_{j=\ell+1}^{n-1}|1-Q(j)| \times\left[\sum_{k=1}^{p}|a_k(\ell)|\sum_{i=\ell-k}^{\ell-1}\left(\sum_{k=1}^{p}|a_k(i)|+\sum_{k=1}^{q}|b_k(i)|\right)\right] + \sum_{\ell=n_0}^{n}\prod_{j=\ell+1}^{n-1}|1-Q(j)| \left[\sum_{k=1}^{q}|b_k(\ell)|\sum_{i=\ell}^{\ell+k-1}\left(\sum_{k=1}^{p}|a_k(i)|+\sum_{k=1}^{q}|b_k(i)|\right)\right]=c<1$$

- $$\liminf_{n\to+\infty}\prod_{j=n_0}^{n-1}|(1-Q(j))|>0$$

If all solution converge to zero then
$$\lim_{n\to+\infty}\prod_{j=n_0}^{n-1}|1-Q(j)|=0$$

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