DIFFERENTIAL SYSTEMS WITH DELAYS

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A differential system

Let us to considerer the differential system

$$x^{(n)}(t) = \sum_{k=1}^{n} A_k x^{(n-k)}(t - \tau_k), \qquad t \geq 0$$

where:

- n is even
- A_k are p-by-p real matrices
- τ_k are positive real numbers, for $k = 1, \dots, n$

The characteristic equation is

$$\lambda^n = \sum_{k=1}^{n} A_k \lambda^{n-k} e^{-\tau_k}$$

A differential equation

It is possible to prove that for

$$M(\lambda) = \sum_{k=1}^{n} A_k \lambda^{n-k} e^{-\tau_k}$$

The equation

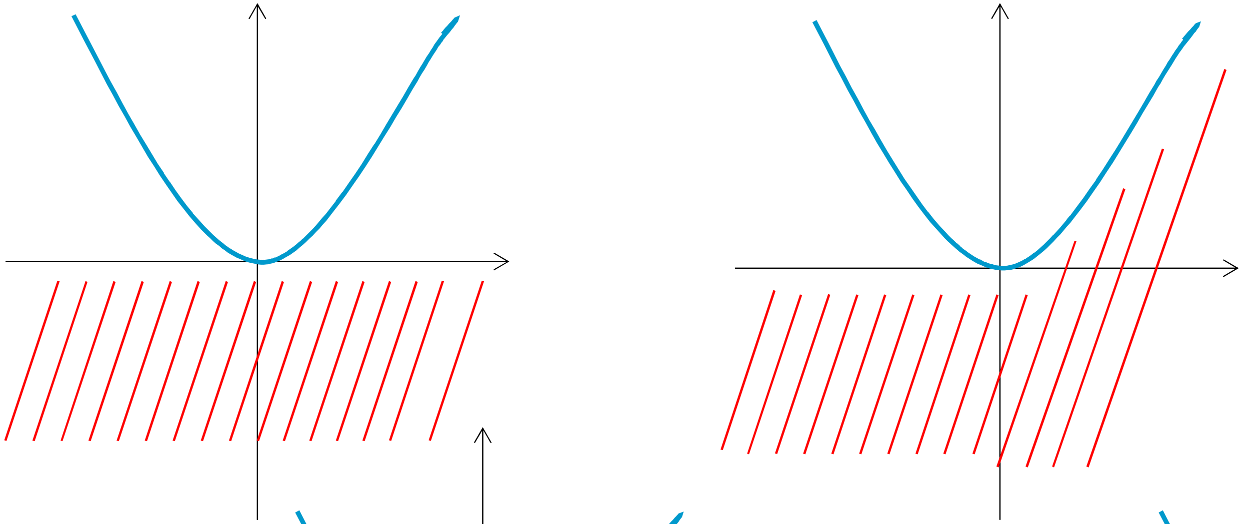
$$x^{(n)}(t) = \sum_{k=1}^{n} A_k x^{(n-k)}(t - \tau_k), \qquad t \geq 0$$

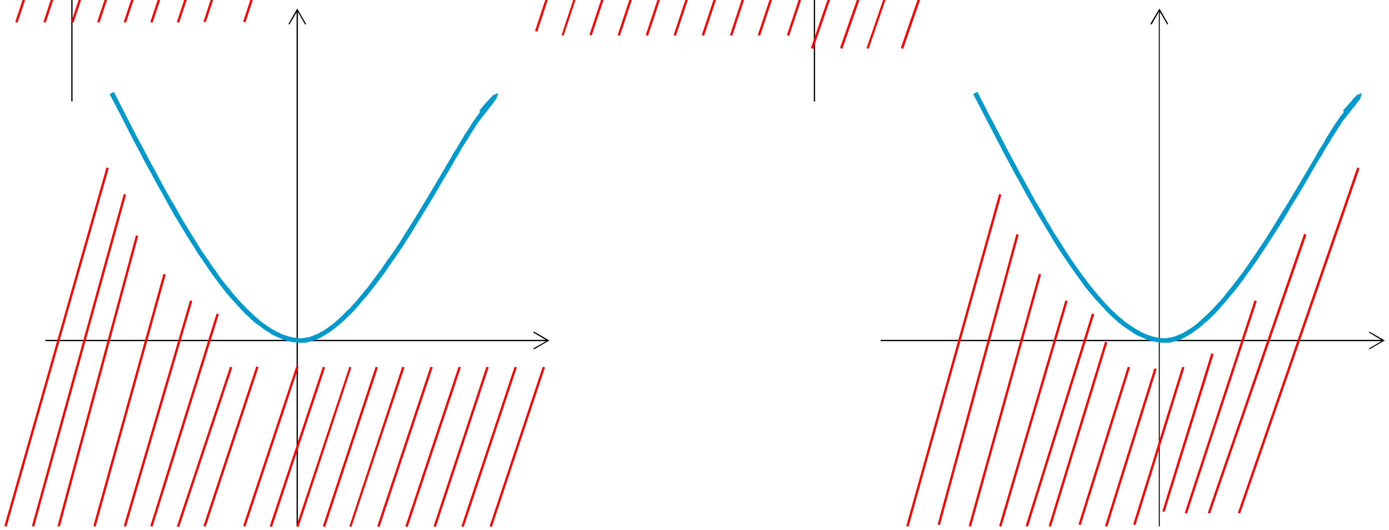
is oscillatory if and only if

$$\lambda^n \notin \left[-\mu\big(-M(\lambda)\big), \mu\big(-M(\lambda)\big)\right]$$

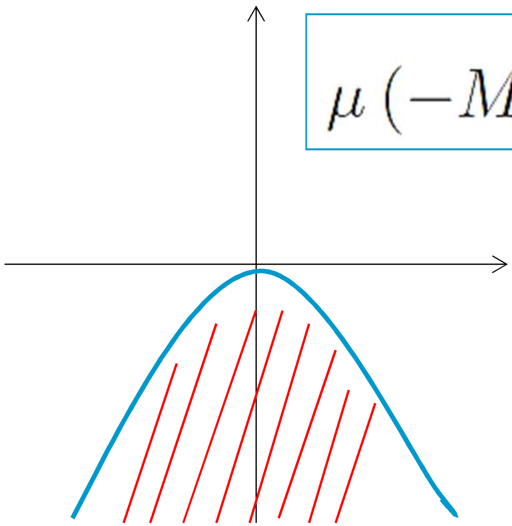
A differential equation

$$\mu\left(M\left(\lambda\right)\right) < \lambda^{n}$$

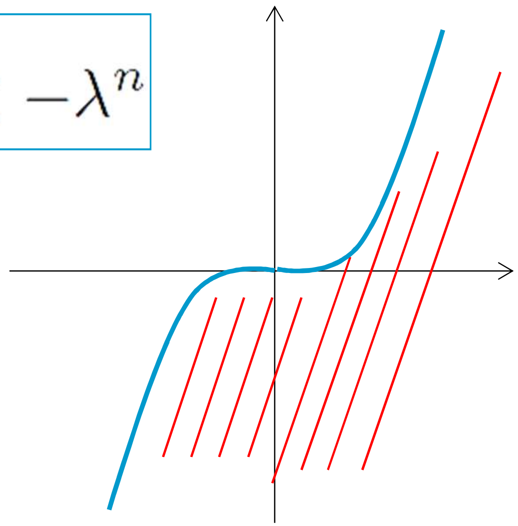




A differential equation



$$\mu\left(-M\left(\lambda\right)\right) < -\lambda^{n}$$



$$\mu\left(M\left(\lambda\right)\right) < \lambda^{n}$$

Oscillatory system

Theorem 1: Let n even. If

- $\mu\left(A_k\right) < 0$
- $\max\limits_{\lambda<0}\left\{\lambda^{-n}e^{-\lambda\tau}\sum\limits_{\substack{k=2\\ k \text{ is evan}}}^{n}\mu\left(A_k\right) - \lambda^{-1}e^{-\lambda T}\sum\limits_{\substack{k=1\\ k \text{ is odd}}}^{n-1}\mu\left(-A_k\right)\right\} < 1$

then

$$x^{(n)}(t) = \sum_{k=1}^{n} A_k x^{(n-k)}(t-\tau_k), \qquad t \geq 0$$

is oscillatory.

Oscillatory system

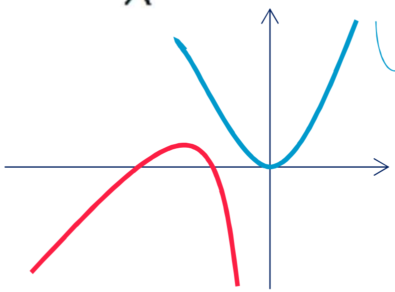
Draft of the proof:

$\lambda = 0 \quad \Longrightarrow \quad \mu\left(M\left(0\right)\right) = \mu\left(A_n\right) < 0$

$\lambda > 0 \quad \Longrightarrow \quad \mu\left(M\left(\lambda\right)\right) \leq \sum_{k=1}^{n} \mu\left(A_k\right) \lambda^{n-k} e^{-\lambda \tau_k} < 0 < \lambda^n$

$\lambda < 0 \quad \Longrightarrow \quad \dfrac{\mu\left(M\left(\lambda\right)\right)}{\lambda^n} < 1$

$$\frac{\mu\left(M\left(\lambda\right)\right)}{\lambda^n} \leq \underbrace{\lambda^{-n} e^{-\lambda \tau} \sum_{\substack{k=2 \\ k \text{ is evan}}}^{n} \mu\left(A_k\right) - \lambda^{-1} e^{-\lambda T} \sum_{\substack{k=1 \\ k \text{ is odd}}}^{n-1} \mu\left(-A_k\right)}_{f\left(\lambda\right)}$$



$$\lim_{\lambda \to 0^-} f\left(\lambda\right) = -\infty = \lim_{\lambda \to -\infty} f\left(\lambda\right)$$

Oscillatory system

Example 1: $x^{(4)}(t) = A_1 x^{(3)}(t-0,1) + A_2 x''(t-\tau_2) + A_3 x'(t-\tau_3) + A_4 x(t-0,4)$

- $0,1 < \tau_2, \tau_3 < 0,4$
- $A_1 = \begin{bmatrix} -8 & 1 \\ 4 & -10 \end{bmatrix} \quad \mu_1(A_1) = \max\{-4,-9\} = -4$
- $A_2 = \begin{bmatrix} -7 & -3 \\ 2 & -4 \end{bmatrix} \quad \mu_1(A_2) = \max\{-5,-1\} = -1$
- $A_3 = \begin{bmatrix} -3 & 4 \\ 0 & -10 \end{bmatrix} \quad \mu_1(A_3) = \max\{-3,-6\} = -3$
- $A_4 = \begin{bmatrix} -5 & 3 \\ 4 & -6 \end{bmatrix} \quad \mu_1(A_4) = \max\{-1,-3\} = -1$

The system is oscillatory

$$-2\lambda^{-4}e^{-0,1\lambda} + 7\lambda^{-1}e^{-0,4\lambda} < -7,673344$$

Oscillatory system

Theorem 2: Let n even. If

- for k even $\mu(A_k) < 0$
- for k odd $\mu(-A_k) < 0$
- $\max\limits_{\lambda>0} \left\{ \lambda^{-n} e^{-\lambda T} \sum\limits_{\substack{k=1 \\ k \text{ is evan}}}^{n} \mu(A_k) - \lambda^{-1} e^{-\lambda \tau} \sum\limits_{\substack{k=1 \\ k \text{ is odd}}}^{n} \mu(A_k) \right\} < 1$

then

$$x^{(n)}(t) = \sum_{k=1}^{n} A_k x^{(n-k)}(t - \tau_k), \qquad t \geq 0$$

is oscillatory.

Oscillatory system

Draft of the proof:

$\lambda = 0 \Longrightarrow \mu(M(0)) = \mu(A_n) < 0$

$\lambda < 0 \Longrightarrow$

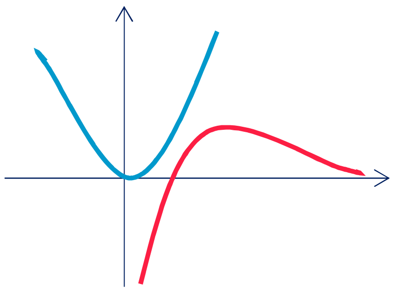
$$\mu(M(\lambda)) \leq \sum_{\substack{k=2 \\ k \text{ is evan}}}^{n} \mu(A_k)\,\lambda^{n-k} e^{-\lambda\tau_k} - \sum_{\substack{k=1 \\ k \text{ is odd}}}^{n-1} \mu(-A_k)\,\lambda^{n-k} e^{-\lambda\tau_k} < 0 < \lambda^n$$

Oscillatory system

Draft of the proof (continuation):

$$\lambda > 0 \quad \longrightarrow \quad \frac{\mu\left(M\left(\lambda\right)\right)}{\lambda^{n}} < 1$$

$$\frac{\mu\left(M\left(\lambda\right)\right)}{\lambda^{n}} \leq \underbrace{\lambda^{-n} e^{-\lambda T} \sum_{\substack{k=2 \\ k \text{ is evan}}}^{n} \mu\left(A_{k}\right) + \lambda^{-1} e^{-\lambda \tau} \sum_{\substack{k=1 \\ k \text{ is odd}}}^{n-1} \mu\left(A_{k}\right)}_{g\left(\lambda\right)}$$



$$\lim_{\lambda \to 0+} g\left(\lambda\right) = -\infty$$

$$\lim_{\lambda \to +\infty} g\left(\lambda\right) = 0$$

Oscillatory system

Example 2: $x^{(4)}(t) = A_1 x^{(3)}(t-1) + A_2 x''(t-\tau_2) + A_3 x'(t-\tau_3) + A_4 x(t-0,25)$

- $0,25 < \tau_2, \tau_3 < 1$
- $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$ $\quad \mu_\infty(A_1) = 1, \quad \mu_\infty(-A_1) = -1/2.$
- $A_2 = \begin{bmatrix} -7 & -3 \\ 2 & -8 \end{bmatrix}$ $\quad \mu_\infty(A_2) = -4$
- $A_3 = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$ $\quad \mu_\infty(A_3) = 1, \quad \mu_\infty(-A_3) = -1/12$
- $A_4 = \begin{bmatrix} -6 & 4 \\ 1 & -9 \end{bmatrix}$ $\quad \mu_\infty(A_4) = -2$

$$-6\lambda^{-4}e^{-\lambda} + 2\lambda^{-1}e^{-\lambda/4} < 0,655185324$$

The system is oscillatory

Nonoscillatory system

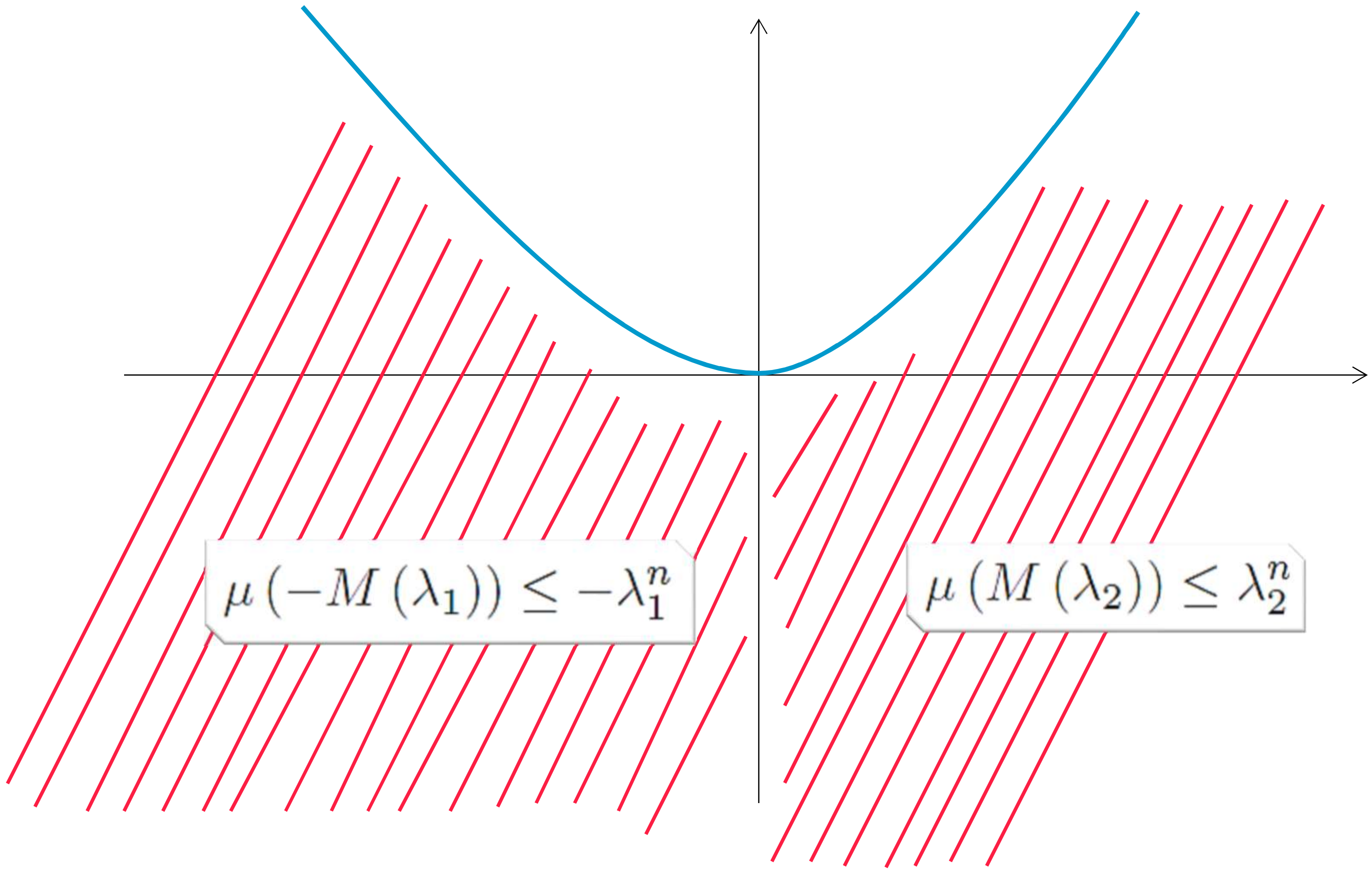
$$\mu(-C) \leq 0 \Rightarrow \det(C) \geq 0;$$
$$if\ d\ is\ odd,\ \mu(C) \leq 0 \Rightarrow \det(C) \leq 0$$

$$x^{(n)}(t) = \sum_{k=1}^{n} A_k x^{(n-k)}(t - \tau_k)$$

is nonoscillatory if and only if there exist a λ_1 and λ_2 such that

$$\mu(-M(\lambda_1)) \leq -\lambda_1^n \quad \text{and} \quad \mu(M(\lambda_2)) \leq \lambda_2^n$$

Nonoscillatory system



Nonoscillatory system

Theorem 4: Let n even. If

- $\mu(-A_k) < 0$ for k = 1, …, n

then

$$x^{(n)}(t) = \sum_{k=1}^{n} A_k x^{(n-k)}(t - \tau_k)$$

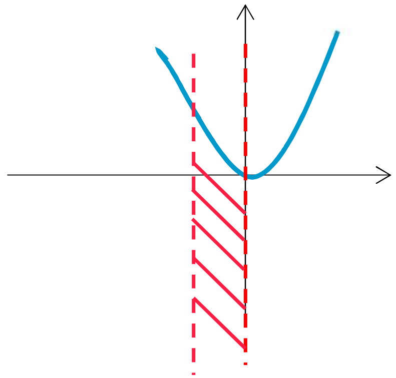
is nonoscillatory independently of the delays.

Nonoscillatory system

Draft of the proof:

- $\lambda < 0$

$$\mu\left(-M\left(\lambda\right)\right) \leq \lambda^n \underbrace{\left(\frac{e^{-\lambda\tau}}{\left(-\lambda\right)^n}\sum_{\substack{k=1\\k\text{ even}}}^{n}\mu\left(-A_k\right)+\frac{e^{-\lambda T}}{-\lambda}\sum_{\substack{k=1\\k\text{ odd}}}^{n}\mu\left(A_k\right)\right)}_{f\left(\lambda\right))}$$



$$\lim_{\lambda\to 0^-} f\left(\lambda\right) = -\infty$$

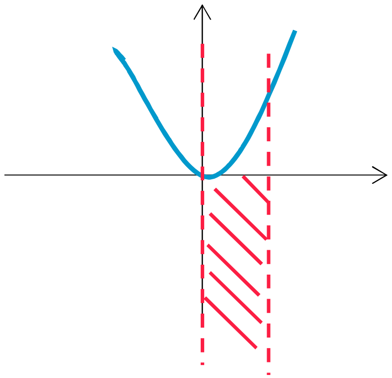
$$f\left(\lambda\right) < -1 \Longrightarrow \mu\left(-M\left(\lambda_1\right)\right) \leq -\lambda_1^n$$

There exist a $\lambda_1 < 0$ such that

Nonoscillatory system

Draft of the proof:

- $\lambda > 0$ $\Longrightarrow$ $$\mu\left(M\left(\lambda\right)\right) \leq \lambda^n \underbrace{\sum_{k=1}^{n} \mu\left(A_k\right) \lambda^{-k} e^{-\lambda \tau_k}}_{h\left(\lambda\right)}$$



$$\lim_{\lambda \to +\infty} h\left(\lambda\right) = 0^+$$

There exist a $\lambda_2 < 0$ such that $h\left(\lambda\right) < 1$ $\Longrightarrow$ $\mu\left(M\left(\lambda_2\right)\right) \leq \lambda_2^n$

Nonoscillatory system

Example 4:

$$x^{(4)}(t) = A_1 x^{(3)}(t-\tau_1) + A_2 x''(t-\tau_2) + A_3 x'(t-\tau_3) + A_4 x(t-\tau_4)$$

$$A_1 = \begin{bmatrix} 6 & 0 & 1 \\ 1 & 4 & -1 \\ 1 & -2 & 7 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 10 & 3 & -2 \\ 0 & 5 & 1 \\ -3 & 1 & 8 \end{bmatrix}$$

$$\mu_\infty(-A_1) = -2$$

The system is nonoscillatory

$$\mu_\infty(-A_2) = -4$$

$$A_3 = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 6 & 0 \\ 0 & 4 & 9 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 8 & 3 & 2 \\ -1 & 4 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mu_\infty(-A_3) = -5$$

$$\mu_\infty(-A_4) = -1$$

Nonoscillatory system

Theorem 5: Let n even.

- for k even $\mu(-A_k) < 0$
- for k odd $\mu(A_k) < 0$

then

$$x^{(n)}(t) = \sum_{k=1}^{n} A_k x^{(n-k)}(t - \tau_k)$$

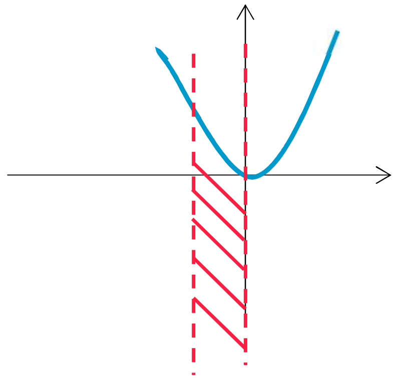
is nonoscillatory independently of the delays.

Nonoscillatory system

Draft of the proof:

- $\lambda < 0$

$$\mu\left(-M(\lambda)\right) \leq \lambda^n \underbrace{\left(\lambda^{-n} e^{-\lambda\tau} \sum_{\substack{k=2 \\ k \text{ even}}}^{n} \mu(-A_k) - \lambda^{-1} e^{-\lambda T} \sum_{\substack{k=1 \\ k \text{ odd}}}^{n-1} \mu(-A_k) \right)}_{f(\lambda))}$$



$$\lim_{\lambda \to 0^-} f(\lambda) = -\infty$$

There exist a $\lambda_1 < 0$ such that

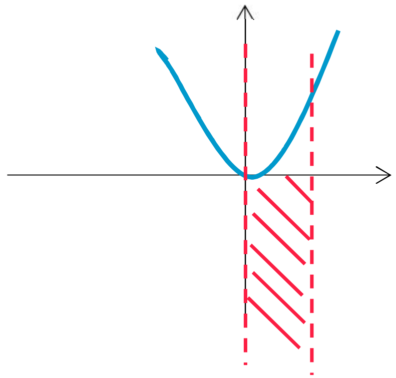
$$f(\lambda) < -1 \Longrightarrow \mu\left(-M(\lambda_1)\right) \leq -\lambda_1^n$$

Nonoscillatory system

Draft of the proof:

- $\lambda > 0$

$$\mu(M(\lambda)) \leq \lambda^n \underbrace{\left(\lambda^{-1}e^{-\lambda\tau}\sum_{\substack{k=2\\ k \text{ even}}}^{n}\mu(A_k) + \lambda^{-n}e^{-\lambda T}\sum_{\substack{k=1\\ k \text{ odd}}}^{n-1}\mu(A_k)\right)}_{h(\lambda)}$$



$$\lim_{\lambda\to+\infty} h(\lambda) = 0^+$$

There exist a $\lambda_2 < 0$ such that $h(\lambda) < 1 \Longrightarrow \mu(M(\lambda_2)) \leq \lambda_2^n$

Nonoscillatory system

Example 5:

$$x^{(4)}(t) = A_1 x^{(3)}(t-\tau_1) + A_2 x''(t-\tau_2) + A_3 x'(t-\tau_3) + A_4 x(t-\tau_4)$$

$$A_1 = \begin{bmatrix} -7 & 1 & -1 \\ 0 & -6 & -1 \\ 3 & 2 & -5 \end{bmatrix}$$

$$\mu_1(A_1) = -3$$

$$A_2 = \begin{bmatrix} 6 & 0 & 3 \\ 1 & -4 & 0 \\ 0 & 2 & 5 \end{bmatrix}$$

$$\mu_1(-A_2) = -2$$

The system is nonoscillatory

$$A_3 = \begin{bmatrix} -9 & 0 & 1 \\ -1 & -1 & 0 \\ 3 & 0 & -2 \end{bmatrix}$$

$$\mu_1(A_3) = -1$$

$$A_4 = \begin{bmatrix} 10 & -3 & 0 \\ 2 & 5 & 3 \\ 0 & 1 & 6 \end{bmatrix}$$

$$\mu_1(-A_4) = -1$$

Nonoscillatory system

Theorem 6: Let n even

- for k = 1, …, i $\quad \mu(-A_k) < 0$
- for k = i+1, …, n $\quad \mu(A_k) < 0$

then

$$x^{(n)}(t) = \sum_{k=1}^{n} A_k x^{(n-k)}(t - \tau_k)$$

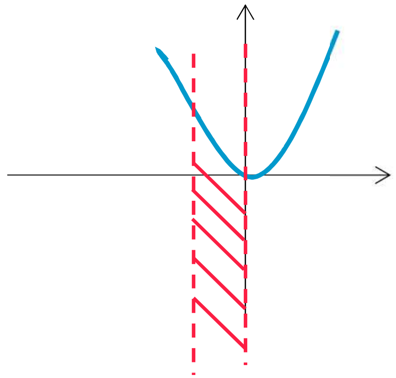
is nonoscillatory independently of the delays.

Nonoscillatory system

Draft of the proof:

$\lambda < 0 \Longrightarrow$

$$\mu\left(-M\left(\lambda\right)\right) \leq \lambda^{n}\Bigg[\underbrace{(-\lambda)^{-2} e^{-\lambda T*} \sum_{\substack{k=2 \\ k \text{ is evan}}}^{i} \mu\left(-A_k\right) + \lambda^{-n} e^{-\lambda \tau *} \sum_{\substack{k=i+1 \\ k \text{ is evan}}}^{n} \mu\left(-A_k\right) - \lambda^{-i} e^{-\lambda \tau} \sum_{\substack{k=1 \\ k \text{ is odd}}}^{i} \mu\left(A_k\right) - \lambda^{-(i+1)} e^{-\lambda T} \sum_{\substack{k=i+1 \\ k \text{ is odd}}}^{n-1} \mu\left(A_k\right)}_{f(\lambda))}\Bigg]$$



$$\lim_{\lambda \to 0^-} f\left(\lambda\right) = -\infty$$

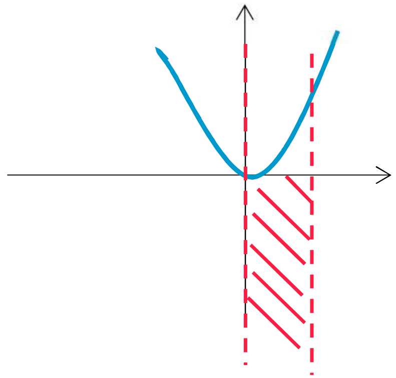
There exist a $\lambda_1 < 0$ such that $f\left(\lambda\right) < -1 \Longrightarrow \mu\left(-M\left(\lambda_1\right)\right) \leq -\lambda_1^n$

Nonoscillatory system

Draft of the proof:

- $\lambda > 0$

$$\mu\left(M\left(\lambda\right)\right) \leq \lambda^{n} \underbrace{\left(\lambda^{-i} e^{-\lambda T} \sum_{k=1}^{i} \mu\left(A_k\right) + \lambda^{-(i+1)} e^{-\lambda \tau} \sum_{k=i+1}^{n} \mu\left(A_k\right)\right)}_{h(\lambda)}$$



$$\lim_{\lambda \to +\infty} h\left(\lambda\right) = 0^{+}$$

There exist a $\lambda_2 < 0$ such that $h\left(\lambda\right) < 1 \Longrightarrow \mu\left(M\left(\lambda_2\right)\right) \leq \lambda_2^{n}$

Nonoscillatory system

Example 6:

$$x^{(6)}(t) = \sum_{k=1}^{5} A_k x^{(n-k)}(t-\tau_k)$$

The system is nonoscillatory

$$A_1 = \begin{bmatrix} -5 & 1 & 0 \\ 3 & -6 & -4 \\ 0 & 2 & -9 \end{bmatrix} \quad A_2 = \begin{bmatrix} -8 & 1 & -2 \\ 1 & -6 & 3 \\ -1 & 0 & -10 \end{bmatrix} \quad \mu_1(A_1) = -2, \quad \mu_1(A_2) = -5$$

$$A_3 = \begin{bmatrix} -8 & 3 & 0 \\ 1 & -4 & 1 \\ 3 & 0 & -2 \end{bmatrix} \quad A_4 = \begin{bmatrix} -3 & 2 & -1 \\ 0 & -5 & 0 \\ 1 & 0 & -6 \end{bmatrix} \quad \mu_1(A_3) = -1, \quad \mu_1(A_4) = -2$$

$$A_5 = \begin{bmatrix} 6 & 4 & -1 \\ -1 & 7 & 0 \\ 2 & 0 & 8 \end{bmatrix} \quad A_6 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & 4 & 2 \\ 1 & 2 & 6 \end{bmatrix} \quad \mu_1(-A_5) = -3, \quad \mu_1(-A_6) = -2$$

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