Equations with delays

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Tunisia, 2022

Introduction

On this lesson we present some of the most basic results in the theory of oscillations of linear delay differential equations with constant and variable coefficients and with constant delays.

Consider the linear delay differential equation

$$x'(t) + \sum_{k=1}^{n} a_k(t)x(t - \tau_k) = 0$$

where:

- $a_k(t) \in \mathrm{C}([\mathrm{t}_0, +\infty), \mathbb{R})$
- $\tau_k \in \mathbb{R}^+ \Longrightarrow \tau = \max\{\tau_1, \tau_2, \ldots, \tau_n\}$

Introduction

A **solution** of the equation

$$x'(t) + \sum_{k=1}^{n} a_k x(t - \tau_k) = 0$$

is a continuous function $x \in \mathrm{C}^1([t_1 - \tau, +\infty), \mathbb{R})$ for some $t_1 \geq t_0$ that verify the equation.

Let an **initial point** t_1 and an **initial function** $\varphi \in \mathrm{C}^1([t_1 - \tau, t_1], \mathbb{R})$.

Then there exist an unique solution x on $[t_1, +\infty)$ such that

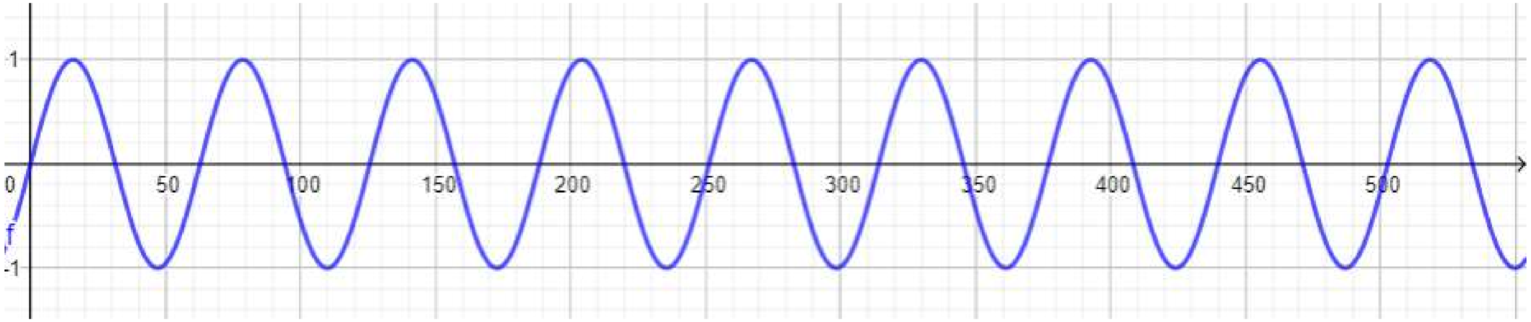
$$x(t) = \varphi(t), \quad \text{for} \quad t_1 - \tau \leq t \leq t_1.$$

The function x is said **oscillatory** if x has arbitrary number of large zeros.

Otherwise is called **non-oscillatory**.

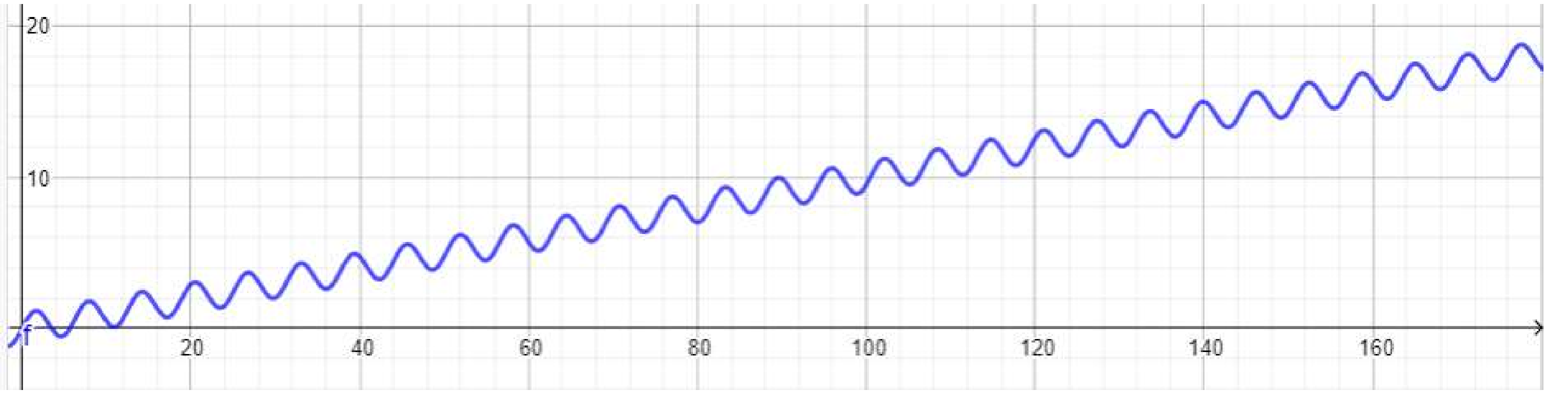
Introduction

Oscillatory



If all solutions od the equation are oscillatory the **equation is oscillatory**

Non-oscillatory



If the equation has at least one non-oscillatory solution then the **equation is non-oscillatoy**

Oscillatory conditions

Consider the linear delay differential equation

$$x'(t) + \sum_{k=1}^{n} a_k\, x(t - \tau_k) = 0$$

Assume $x(t) = e^{\lambda t}$ as a solution of the equation.

Then

$$\lambda e^{\lambda t} + \sum_{k=1}^{n} a_i\, e^{\lambda(t-\tau_k)} = 0$$

$$\lambda + \sum_{k=1}^{n} a_i\; e^{-\lambda \tau_k} = 0 \quad \Longrightarrow \quad \textbf{Characteristic solution}$$

Oscillatory conditions

Theorem 1

- $a_i \in \mathbb{R}$
- $\tau_i \in \mathbb{R}^+$

Then every solutions of

$$x'(t) + \sum_{k=1}^{n} a_k x(t - \tau_k) = 0$$

oscillate if and only if the characteristic equation

$$\lambda + \sum_{k=1}^{n} a_k \; e^{-\lambda \tau_k} = 0$$

has no real roots.

Oscillatory conditions

Proof (draft):

It is evident that if every solutions of

$$x'(t) + \sum_{i=1}^{n} a_i(t)x(t - \tau_i) = 0$$

oscillate then the characteristic equation

$$\lambda + \sum_{i=1}^{n} a_i \ e^{-\lambda \tau_i} = 0$$

has no real roots.

Notice that if the characteristic equation has a real root then $e^{\lambda t}$ is a non-oscillatory solution.

Oscillatory conditions

Proof (draft):

Let us to proof that if characteristic equation

$$\lambda + \sum_{i=1}^{n} a_i \ e^{-\lambda \tau_i} = 0$$

has no real roots then every solutions of

$$x'(t) + \sum_{i=1}^{n} a_i(t) x(t - \tau_i) = 0$$

oscillate.

For a sake of contradiction we assume that the characteristic solution has no real roots and there exist a positive solution $x(t)$.

Oscillatory conditions

Proof (draft):

It is possible to prove that there exist constants M and μ such that

$$|x(t)| \leq Me^{\mu t}$$

So, the Laplace transform

$$X(s) = \int_0^{+\infty} e^{-s}\, x(t)dt$$

exist to $Re(s) > \mu$.

So,

$$\int_0^{+\infty} e^{-st} x'(t)dt = sX(s) - x(0)$$

$$\left(s + \sum_{i=1}^{n} a_i\ e^{-s\tau_i}\right) X(s) = x(0) - \sum_{i=1}^{n} a_i\ e^{-\lambda s} \int_{-\tau_k}^{0} e^{-st} x(t)dt$$

$$\int_0^{+\infty} e^{-st} x(t - \tau_k)dt = e^{-s\tau_k} X(s) + e^{-st} \int_{-\tau_k}^{0} e^{-s}\, x(t)dt$$

Oscillatory conditions

Contradiction!

Proof (draft):

$$\left(s+\sum_{i=1}^{n} a_i\ e^{-s\tau_i}\right)X(s)=x(0)-\sum_{i=1}^{n} a_i\ e^{-\lambda s}\int_{-\tau_k}^{0} e^{-st}x(t)dt$$

$$X(s)=\frac{x(0)-\sum_{i=1}^{n} a_i\ e^{-\lambda s}\int_{-\tau_k}^{0} e^{-st}x(t)dt}{s+\sum_{i=1}^{n} a_i\ e^{-s\tau_i}}$$

Negative

Positive

Positive

Oscillatory conditions

Theorem 2

- $a \in \mathbb{R}$
- $\tau \in \mathbb{R}^{+}$

Then every solutions of

$$x'(t) + ax(t-\tau) = 0$$

oscillate if and only if

$$a\tau > 1/e$$

Oscillatory conditions

Proof (draft):

It is enough to prove that to show that $a\tau > 1/e$ is equivalent to the statement

“the characteristic equation

$$\lambda + ae^{-\lambda\tau} = 0$$

has no real root”.

Let

$$\mathrm{F}(\lambda) = \lambda + ae^{-\lambda\tau}$$

The characteristic equation has no real roots!

Notice that we must have

$$a\tau > 0 \Longrightarrow a > 0$$

So

$$\mathrm{F}(-\infty) = +\infty = \mathrm{F}(+\infty)$$

$$\min \mathrm{F}(\lambda) = \frac{\ln(p\lambda e)}{\tau} > 0$$

Oscillatory conditions

Theorem 3

- $a_i \in \mathbb{R}^+$ and $\tau_i \in \mathbb{R}^+$

Then each of the following two conditions is sufficient for the oscillations of all solutions of equation

$$x'(t) + \sum_{k=1}^{n} a_k x(t - \tau_k) = 0$$

a)

$$\sum_{k=1}^{n} a_k \ \tau_k > 1/e$$

b)

$$\left(\prod_{k=1}^{n} a_i\right)^{\frac{1}{n}} \left(\sum_{k=1}^{n} \tau_k\right) > e$$

Oscillatory conditions

$$x'(t) + \sum_{k=1}^{n} a_k x(t - \tau_k) = 0$$

Theorem 4

- $a_i \in \mathbb{R}^+$ and $\tau_i \in \mathbb{R}^+$

Then

$$\left(\sum_{k=1}^{n} a_k\right)\left(\max_{1 \le k \le n} \tau_k\right) \le 1/e$$

is a sufficient condition for the existence of a non-oscillatory solution of equation

And

$$\left(\sum_{k=1}^{n} a_k\right)\left(\min_{1 \le k \le n} \tau_k\right) > 1/e$$

is a sufficient condition for the existence of a non-oscillatory solution of equation

Generalized characteristic equation

Let

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where:

- $a_k(t) \in \mathrm{C}([t_0, +\infty), \mathbb{R})$
- $\tau_k \in$ [illegible] $\tau = \max\{\tau_1, \tau_2, \dots, \tau_n\}$

where

- $a_k(t) \in \mathrm{C}$
- $\tau_k \in \mathrm{C}([t_0$

with the initi

and $\varphi(t) \in \mathrm{C}([t_{-1}, t_0], \mathbb{R})$.

Generalized characteristic equation

Theorem 5: The following statement are equivalent:

a) The solution of

$$x'(t) + \sum_{k=1}^{n} a_k(t)x\big(t - \tau_k(t)\big) = 0, \qquad t_0 \le t \le T$$

$$x(t) = \varphi(t), > 0 \qquad t_{-1} \le t \le t_0$$

is positive on $t_0 \le t \le T$

$h_k(t) = \min\{t_0, t - \tau_k(t)\}$

b) The generalised equation

$$\alpha(t) + \sum_{k=1}^{n} a_k(t) \frac{\varphi\big(h_k(t)\big)}{\varphi(t_0)} e^{-\int_{H_k(t)}^{t} \alpha(s)ds} = 0$$

$H_k(t) = \max\{t_0, t - \tau_k(t)\}$

has a continuous solution on $t_0 \le t \le T$

Comparison results

Let

$$x'(t) + \sum_{k=1}^{n} a_k(t) x\big(t - \tau_k(t)\big) = 0$$

and

$$y'(t) + \sum_{k=1}^{n} b_k(t) y\big(t - \tau_k(t)\big) \leq 0$$

$$z'(t) + \sum_{k=1}^{n} c_k(t) z\big(t - \tau_k(t)\big) \geq 0$$

where:

- $a_k(t), b_k(t), c_k(t), \tau_k(t) \in \mathrm{C}([\mathrm{t}_0, T), \mathbb{R})$

Comparison results

$$x'(t) + \sum_{k=1}^{n} a_k(t)x\big(t - \tau_k(t)\big) = 0$$

$$y'(t) + \sum_{k=1}^{n} b_k(t)y\big(t - \tau_k(t)\big) \le 0$$

$$z'(t) + \sum_{k=1}^{n} c_k(t)z\big(t - \tau_k(t)\big) \ge 0$$

Theorem 6: If

- $0 < c_k(t) \le a_k(t) \le b_k(t)$
- $x(t), y(t)$ and $z(t)$ are continuous solutions such that
  - $y(t) > 0, \quad t_0 \le t < T$
  - $y(t_0) \le x(t_0) \le z(t_0)$
  - $0 \le \frac{z(t)}{z(t_0)} \le \frac{x(t)}{x(t_0)} \le \frac{y(t)}{y(t_0)}, \qquad t_1 \le t < t_0$

Then

$$y(t) \le x(t) \le z(t), \qquad t_0 \le t < T.$$

Comparison results

Theorem 7: If

- $a_k(t) > 0$
- $\varphi(t), \psi(t) \in C([t_{-1}, t_0], \mathbb{R})$ such that
  - $0 < \varphi(t_0) \leq \psi(t_0)$
  - $0 \leq \frac{\psi(t)}{\psi(t_0)} \leq \frac{\varphi(t)}{\varphi(t_0)}, \quad t_1 \leq t < t_0$
- $x(\varphi)(t) > 0, \quad t_0 \leq t < T$

Then

$$x(\varphi)(t) \leq x(\psi)(t), \qquad t_0 \leq t < T.$$

$$x'(t) + \sum_{k=1}^{n} a_k(t) x\big(t - \tau_k(t)\big) = 0$$

Linearized oscillation theory

Consider the non-linear delay differential equations

$$x'(t) + \sum_{k=1}^{n} a_k f\big(x(t-\tau_k)\big) = 0$$

where:

- $a_k \in \mathbb{R}^+$
- $\tau_k \in \mathbb{R}^+$
- $f_k \in C(\mathbb{R}, \mathbb{R})$
- $u f_k(u) > 0$

If

$$\liminf_{u \to 0} \frac{f_k(u)}{u} \geq 1 \quad \text{and} \quad \lim_{u \to 0} \frac{f_k(u)}{u} = 1$$

Then the equation

$$y'(t) + \sum_{k=1}^{n} a_k y(t-\tau_k) = 0$$

is the **linearized equation** associated with

$$x'(t) + \sum_{k=1}^{n} a_k f\big(x(t-\tau_k)\big) = 0$$

Linearized oscillation theory

The delay effect: Nichoson's blowflies

The delay differential equation

$$N'(t) = -\delta N(t) + PN(t-\tau)e^{-aN(t-\tau)}.$$

was used by Gurney, S. P. Blythe and R. M. Nishbet (1980)

and it was used to describe the dynamics of Nicholson's blowflies.

Here:

- P is the maximum per capita daily egg production rate
- $1/a$ is the size at which the population reproduces at its maximum rate
- δ is the per capita daily adult death rate

$-\tau)$

$$\lim_{t\to+\infty} Q(t) = \delta\left(\ln\left(\frac{}{\delta}\right) - 1\right) = R$$

$$z'(t) + \delta x(t) + Rz(t-\tau) = 0$$

$$\lambda + \delta + Re^{-\lambda} = 0$$

is the characteristic equation

Linearized oscillation theory

$$N'(t) = rN(t)\left(1 - \frac{N(t-\tau)}{K}\right) \Longrightarrow$$

Generalization

$$N(t) = Ke^{x(t)}$$

$$x'(t) + r\frac{e^{x(t-\tau)} - 1}{1 + cre^{x(t-\tau)}} = 0$$

$$x'(t) + \frac{r}{1+cr} f\big(x(t-\tau)\big) = 0$$

$$y'(t) + \frac{r}{1+cr} y(t-\tau) = 0$$

Every positive solutions of generalized logistic equation oscillate about the steady state K if and only if

$$\frac{r}{1+cr}\tau > \frac{1}{e}$$

Linearized oscillation theory

The delay effect: survival of red blood cells

The delay differential equation

$$N'(t) = -\mu N(t) + pe^{-\gamma N(t-\tau)}.$$

has been used by Wazewska-Czyzewska and Lasota (1988)

as a model for the survival of red blood cells in an animal.

Here:

- μ is the probability of death of a red blood cell
- p and γ are positive constants and are related to the production of red blood cells per unit of time
- τ is the time required to produce a red blood cell.

The sol

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$x(t)$ oscillate about zero

Difference equations

Consider the linear delay difference equation

$$\Delta x(n) + \sum_{i=1}^{m} p_i x(n - k_i) = 0$$

where:

- $p_i \in \mathbb{R}^+$
- $k_i \in \mathbb{N}_0$

The characteristic equation

$$\boldsymbol{\lambda - 1 + \sum_{i=1}^{m} p_i \lambda^{-k_i} = 0}$$

Difference equations

$$x'(t) + \sum_{k=1}^{n} a_k x(t - \tau_k) = 0 \quad \Longrightarrow \quad \boldsymbol{\lambda + \sum_{k=1}^{n} a_i \, e^{-\lambda \tau_k} = 0}$$

$$x(t) = e^{\lambda t}$$

$$\Big\Downarrow \quad e^{\lambda t} = \left(e^{\lambda}\right)^{t} = \mu^{n}$$

$$x(n) = \mu^{n}$$

$$\Delta x(n) + \sum_{i=1}^{m} p_i x(n - k_i) = 0 \quad \Longrightarrow \quad \boldsymbol{\mu - 1 + \sum_{i=1}^{m} p_i \mu^{-k_i} = 0}$$

Theorem 8: Every solutions of the difference equation oscillate if and only if the characteristic equation has no positive roots

Difference equati

Theorem 9: Suppose that

$$\sum_{i=1}^{m} p_i \frac{(}{}$$

Then all solutions of

$$\Delta x(n) + \sum_{i=1}^{m} p_i x$$

oscillates.

$$\sum_{i=1}^{m} p_i \frac{(k_i+1)^{k_i+1}}{k_i^{k_i}} = \sum_{i=1}^{m} p_i \left(1+\frac{1}{k_i}\right)^{k_i} (k_i+1) > 1$$

Oscillatory conditions

Theorem 3

- $a_i \in \mathbb{R}^+$ and $\tau_i \in \mathbb{R}^+$

Then each of the following two conditions is sufficient for the oscillations of all solutions of equation

$$x'(t) + \sum_{k=1}^{n} a_k x(t-\tau_k) = 0$$

a)

$$\sum_{k=1}^{n} a_k \ \tau_k > 1/e$$

b)

$$\left(\prod_{k=1}^{n} a_i\right)^{\frac{1}{n}} \left(\sum_{k=1}^{n} \tau_k\right) > e$$

Linearized oscillation theory

Consider the non-linear difference equation

$$\Delta x(n) + \sum_{i=1}^{m} p_i f_i\big(x(n-k_i)\big) = 0$$

where

- $p_i \in \mathbb{R}^+$
- $k_i \in \mathbb{N}_0$
- $f_i \in C(\mathbb{R}, \mathbb{R})$ and $u f_i(u) > 0$

If

$$\liminf_{u\to 0} \frac{f_i(u)}{u} \geq 1 \quad \text{and} \quad \lim_{u\to 0} \frac{f_i(u)}{u} = 1$$

Then the equation

$$\Delta y(n) + \sum_{k=1}^{n} p_i y(n-k_i) = 0$$

is the **linearized equation** associated with

$$\Delta x(n) + \sum_{i=1}^{m} p_i f_i\big(x(n-k_i)\big) = 0$$

References

📖 Gyori, I and Ladas, G., **Oscillation Theory of Delay Differential Equations**, Mathematical Monographs, Oxford University Press, New York (1991).

… and references therein!