OSCILLATIONS FOR DELAY SYSTEMS

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Tunisia, 2022

Introduction

$$x(t) = \int_{-1}^{0} x(t - r(\theta))\, d[\nu(\theta)]$$

- $x(t) \in \mathbb{R}^d$,

- $r(\theta)$ is a real positive continuous function on $[-1,0]$,

- $\nu(\theta)$ is a matrix of bounded variation.

Introduction

The initial value problem we define

$$x(t) = \int_{-1}^{0} x(t - r(\theta))\, d[\nu(\theta)]\,, \qquad t \geq 0$$

$$x(t) \ = \ \phi(t), \qquad t\ [-\|r\|, 0]$$

- $\|r\| \ = \max_{\theta \in [-1,0]} \ r(\theta),$
- $\phi(t) \in C([-\|r\|, 0], \mathbb{R}^d)$
- $\phi(0) = \int_{-1}^{0} \phi(-r(\theta)) d[\nu(\theta)]$

✓ Exist only one solution x(t) for the initial value problem.

✓ x(t) is exponentially bounded

Oscillatory behaviour

In a interval $]a, +\infty[$, we will say a function satisfies (C) **frequently** or **persistently** whenever for every $t_0 > a$ there exists a $t > t_0$ such that $f(t)$ verifies (C).

On the contrary, if there exists a $t_0 > a$ such that $f(t)$ verifies (C) for every $t > t_0$, it is said to satisfy (C) **eventually** or **ultimately**.

Example:

a) $f(x) \to a$ if $\forall \varepsilon > 0$: $|f(x) - a| < \varepsilon$ eventually;

b) $f(x) \nrightarrow a$ if $\exists \varepsilon > 0$: $|f(x) - a| \geq \varepsilon$ frequently.

Oscillatory behaviour

- A function $f(t) = [f_1(t), \ldots, f_d(t)]^T$ is said **oscillatory componentwise** if each function $f_k(t)$ is frequently nonnegative and frequently nonpositive.
  - If for some $k \in \{1, \ldots d\}$ exist $f_k(t)$ is either eventually positive or eventually negative, $f(t)$ is said a **nonoscillatory componentwise** function.

- A function $f(t) = [f_1(t), \ldots, f_d(t)]^T$ is said **weakly oscillatory** if there exist a $k \in \{1, \ldots, d\}$ such that the function $f_k(t)$ is frequently nonnegative and frequently nonpositive.
  - If for each $k \in \{1, \ldots, d\}$ the function $f_k(t)$ is either eventually positive or eventually negative, is said a **weakly nonoscillatory** function .

Oscillatory behaviour

The system is:

- **oscillatory componentwise** if all solutions are **oscillatory componentwise**

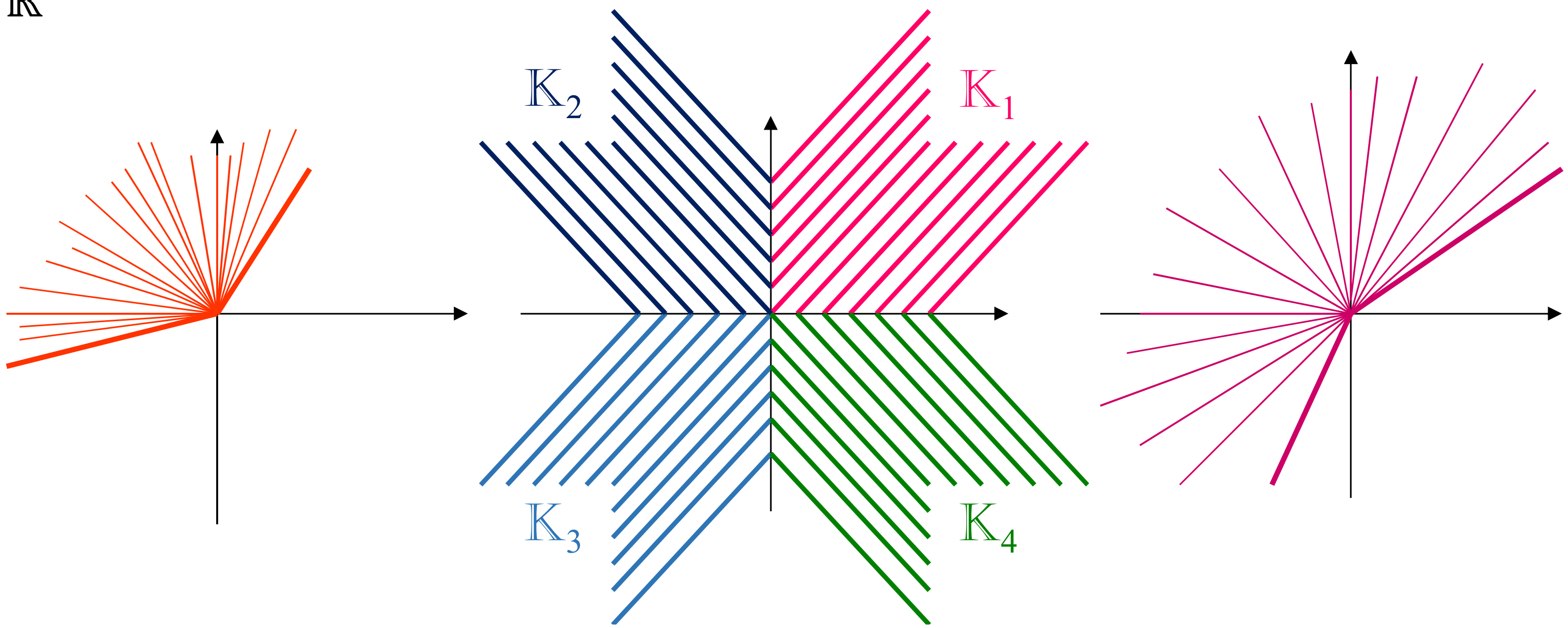
- **weakly oscillatory** if all solutions are **weakly oscillatory** .

Oscillatory behaviour

- A set $K \in R^n$ is a **cone** if for each $u, v \in K, u \neq 0$, and $a > 0, b > 0$ we have

$$au + bv \in K \quad \text{and} \quad -u \notin K.$$

- **Example:** In $\mathbb{R}^2$



Oscillatory behaviour

- Considering $r_M = \max\{r(\theta)\colon -1 \leq \theta \leq 0\}$, a continuous function $x\colon [-r_M, +\infty[\to \mathbb{R}$, is

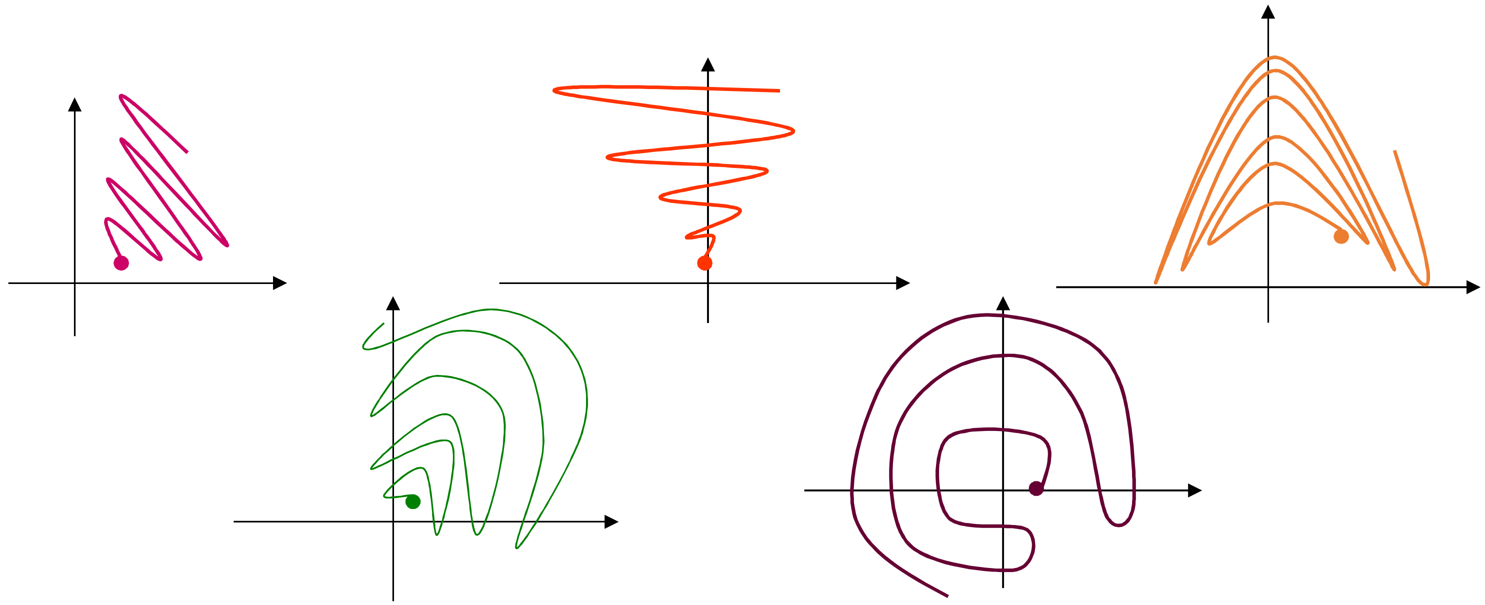
  K-nonoscillatory if there is $T \geq 0$ and a closed cone $K \in \mathbb{R}^n$ such that

$$x(t) \in K\{0\} \text{ for all } t{\geq}T.$$

- Otherwise, it is called **K-oscillatory**.

- Whenever all solutions are K-oscillatory we will say that the system is an **K-oscillatory system**.

Oscillatory behaviour



- weakly nonoscillatory $\Rightarrow$ nonoscillatory (K)
- weakly nonoscillatory $\Rightarrow$ nonoscillatory componentewise

Oscillatory behaviour

An equation is

oscillatory componentwise

weakly oscillatory

K**-oscillatory**

if the characteristic equation has no real zeros.

Characteristic equation

Theorem: Every solutions of

$$x(t) = \int_{-1}^{0} x(t - r(\theta))\, d[\nu(\theta)]$$

oscillate if and only if the characteristic equation

$$\det\left(I - \int_{-1}^{0} e^{-\lambda r(\theta)} d[\nu(\theta)]\right) \neq 0$$

$\forall \lambda \in \mathbb{R}$.

Logarithmic norm

Each induced norm, $\|\cdot\|$, on $M_d(\mathbb{R})$, we associate a logarithmic norm,

$$\mu(C) = \lim_{\gamma \to 0+} \frac{\|I + \gamma C\| - 1}{\gamma}$$

$\Rightarrow$ $$\mu_1(C) = \max_{1 \leqslant k \leqslant n} \left\{ c_{kk} + \sum_{j \neq k} |c_{jk}| \right\}$$

$\Rightarrow$ $$\mu_\infty(C) = \max_{1 \leqslant j \leqslant n} \left\{ c_{jj} + \sum_{k \neq j} |c_{jk}| \right\}$$

Example: Let the matrix $C = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 3 \\ -3 & 0 & 2 \end{bmatrix}$

- $\mu_1(C) = \max\{-1 + 0 + |-3|,\ \ 0 + (-1) + 0,\ \ |-1| + 3 + 2\} = 6$
- $\mu_\infty(C) = \max\{-1 + 0 + |-1|,\ \ 0 - 1 + 3, |-3| + 0 + 2\} = 5.$

Logarithmic norm

For a finite sequence, $C_1, \dots, C_\ell$, on $M_d(IR)$, we define the matrix measure

$$a(C_k) = \mu\left(\sum_{i=1}^{k} C_i\right) \qquad\qquad b(C_k) = \mu\left(\sum_{i=k}^{\nu} C_i\right)$$

$$\alpha(C_1) = a(C_1) = \mu(C_1) \qquad\qquad \beta(C_\nu) = b(C\nu) = \mu(C_\nu)$$

$$\alpha(C_k) = a(C_k) - a(C_{k-1}) \qquad\qquad \beta(C_k) = b(C_k) - b(C_{k+1})$$

See: J. Kirchner and U. Stroinsky, Explicit oscillation criteria for systems of neutral equations with distributed delay. Differential Equations and Dynam. Systems 3: 101-120 (1995)

Logarithmic norm

- $Re\,\sigma(C) \in [-\mu(-C), \mu(C)]$, with $\sigma(C) = \{\lambda\colon \lambda\ eigenvalue\ C\}$;
- $s(C) \ \leq \ \mu(C) \ \leq \ \|C\|$, with $s(C) = \sup\{Re\lambda_i : \lambda_i\ eigenvalue\ C\}$;
- $\mu(C_1) - \mu(-C_2) \ \leq \ \mu(C_1 + \ C_2) \ \leq \ \mu(C_1) + \mu(C_2)$;
- $\mu(\gamma C) \ = \ \gamma\mu(C)$, for every $\gamma \geq 0$;
- $\mu(-C) \leq 0 \ \Rightarrow \ \det(C) \geq 0$;
- $\mu(C) \leq 0 \ \Rightarrow \ \det(C) \leq 0$ if d odd;
- $\mu(C) \leq 0 \ \Rightarrow \ \det(C) \geq 0$ if d even;

Logarithmic norm

- if $\varphi \in C([a,b];\mathbb{R})$ is nonincreasing and positive, then

$$\mu\left(\int_a^b \varphi(\theta)d[\eta(\theta)]\right) \leq \int_a^b \varphi(\theta)d\mu\big(\eta(\theta)-\eta(a)\big)$$

- if $\varphi \in C([a,b];\mathbb{R})$ is nondecreasing and positive, then

$$\mu\left(\int_a^b \varphi(\theta)d[\eta(\theta)]\right) \leq -\int_a^b \varphi(\theta)d\mu\big(\eta(b)-\eta(\theta)\big)$$

Oscillations

It is possible to prove that for

$$M(\lambda) = \int_{-1}^{0} e^{-\lambda r(\theta)} d[\nu(\theta)],$$

the equation

$$x(t) = \int_{-1}^{0} x(t - r(\theta))\, d[\nu(\theta)]$$

is oscillatory if and only if

$$1 \notin \left[-\mu\big(-M(\lambda)\big), \mu\big(M(\lambda)\big)\right]$$

Some results

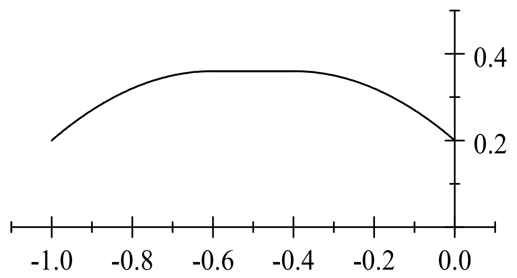
↳ If

- $\mu(\nu(\theta)-\nu(-1))$ is nonincreasing and $\mu(\nu(0)-\nu(\theta))$ is nondecreasing

then the delay system is oscillatory independently of the monotonic delays on $[-1,0]$.

↳ Let $r(\theta)$ a function on $D^{+}(\theta_1,\theta_2)$.

- $\mu\big(\nu\,(\theta_2)-\nu(\theta_1)\big)\leq 0$,
- $\mu\big(\nu(\theta_1)-\nu(\theta)\big)\leq 0,\;\; \mu\big(\nu(\theta)-\nu(-1)\big)\geq 0$, for $\theta\in[-1,\theta_1]$,
- $\mu\big(\nu(\theta)-\nu\,(\theta_2)\big)\leq 0,\;\; \mu\big(\nu\,(0)-\nu(\theta)\big)\geq 0$, for $\theta\in[\theta_2,0]$,



and

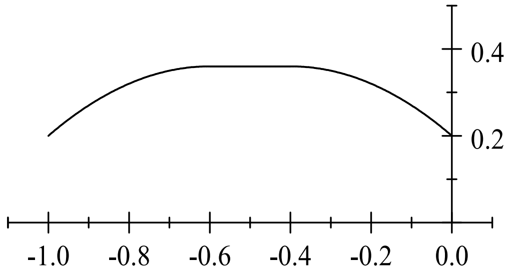
$$\int_{-1}^{0}\mu\big(\nu(\theta)-\nu(-1)\big)\,d\ln\big(r(\theta)\big)-\int_{-1}^{0}\mu\big(\nu(0)-\nu(\theta)\big)\,d\ln\big(r(\theta)\big)<e$$

then the delay system is oscillatory.

Some results

If

- $\mu\big(\nu(-1)-\nu(\theta)\big)$ is nondecreasing on $[-1,\theta_1]$,
- $\mu\big(\nu(\theta)-\nu(0)\big)$ is nonincreasing on $[\theta_2,0]$,



and

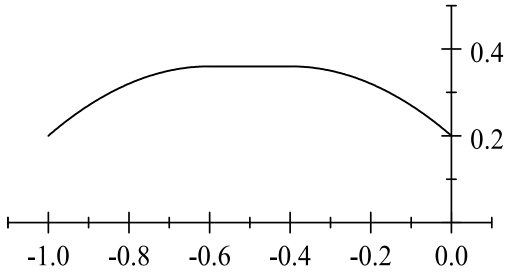
$$\mu\big(\nu(-1)-\nu(\theta_1)\big)+\mu\big(\nu(\theta_1)-\nu(\theta_2)\big)+\mu\big(\nu(\theta_2)-\nu(0)\big)\leq -1,$$

then the delay system is nonoscillatory for every delays on $D^+(\theta_1,\theta_2)$.

Some results

↳ If

- $\mu\big(\nu(\theta) - \nu(\theta_1)\big)$ is nonincreasing on $[-1, \theta_1]$,
- $\mu\big(\nu(\theta_2) - \nu(\theta)\big)$ is nondecreasing on $[\theta_2, 0]$,



and

$$\mu\big(\nu(-1) - \nu(\theta_1)\big) + \mu\big(\nu(\theta_1) - \nu(\theta_2)\big) + \mu\big(\nu(\theta_2) - \nu(0)\big) \leq -1$$

then the delay system is nonoscillatory for every delays on $D^+(\theta_1, \theta_2)$.

A particular case

The delay system

$$x(t) = \int_{-1}^{0} x(t - r(\theta))\, d[\nu(\theta)]$$

include the important class the delay difference systems

$$x(t) = \sum_{k=1}^{\ell} P_k x(t - r_k)$$

when

- $\nu(\theta) = \sum_{k=1}^{\ell} H(\theta - \theta_k) P_k$, where H is the Heaviside function,
- $-1 < \theta_1 < \cdots < \theta_i < 0$,
- $r(\theta)$ is a continuous and positive function on $[-1,0]$ such that $r(\theta_k) = r_k$, for $k = 1, \ldots, \ell$.

A particular case

The characteristic equation of the delay difference systems

$$x(t) = \sum_{k=1}^{\ell} P_k x(t - r_k)$$

is given by

$$\det\left(I\lambda - \sum_{k=1}^{\ell} P_k e^{-\lambda r_k} \right) = 0$$

and the system is oscillatory is and only if the characteristic equation has no real roots.

Some results

- If
  - $\mu(P_i) \;\leq\; 0$, for every $j = 1, \dots, \ell$,

then the difference system is oscillatory for every monotonic families of the delays $(r_1, \dots, r_\ell) \in \mathbb{R}_+^\ell$.

$$\alpha(P_i) = \mu\left(\sum_{k=1}^{i} P_k\right) - \mu\left(\sum_{k=1}^{i-1} P_k\right)$$

- If for $i \;= 1, \dots, \ell$,
  - $\alpha(P_i) \leq 0$
  - $\beta(P_i) \leq 0$,

$$\beta(P_i) = \mu\left(\sum_{k=i}^{\ell} P_k\right) - \mu\left(\sum_{k=i+1}^{\ell} P_k\right)$$

then the difference system is oscillatory for every monotonic families of the delays $(r_1, \dots, r_\ell) \in \mathbb{R}_+^\ell$.

Some results

Let

- $r_1 > r_2 > \cdots > r_\ell$,
- $a(Pi) \leq 0$
- $b(Pi) \geq 0$

$$a(P_i) = \mu\left(\sum_{k=1}^{i} P_k\right)$$

$$\beta(P_i) = \mu\left(\sum_{k=i}^{\ell} P_k\right)$$

Then the difference system is oscillatory if

$$b(P_i)\ln\left(\frac{r_i}{r_{i-1}}\right) > -e.$$

Some results

Let

- $r_1 < r_2 < \cdots < r_\ell$,
- $a(P_i) \geq 0$
- $b(Pi) \leq 0$.

Then the difference system is oscillatory if

$$a(P_i) \ln\left(\frac{r_{i+1}}{r_i}\right) < e.$$

An example

$$x(t) = P_1 x\left(t - \frac{7}{4}\right) + P_2 x\left(t - \frac{3}{2}\right) + P_3 x\left(t - \frac{5}{4}\right)$$

with

$$P_1 = \begin{bmatrix} -3 & -5 \\ 1 & -9 \end{bmatrix} \qquad P_2 = \begin{bmatrix} 1 & 2 \\ -2 & -5 \end{bmatrix} \qquad P_3 = \begin{bmatrix} -1 & 1 \\ 4 & -3 \end{bmatrix}$$

We have:

- $a(P_1) = \mu_1(P_1) = -2,$
- $a(P_2) = \mu_1(P_1 + P_2) = -1,$
- $a(P_3) = b(P_1) = \mu_1(P_1 + P_2 + P_3) = 0$
- $b(P_2) = \mu_1(P_2 + P_3) = 5$
- $b(P_3) = \mu_{1(P_3)} = 3$
- $b(P_2) \ln\left(\frac{r_2}{r_1}\right) + b(P_3) \ln\left(\frac{r_3}{r_2}\right) \approx -1{,}32 > -e$

Then the difference system is oscillatory

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