Difference vs Differential Equations

Oscillatory and Asymptotic Behavior

Sandra Pinelas

Tunisia, 2022

Introduction

The first part of this work is dedicated at the oscillatory behavior of the mixed type difference equation with variable coefficients

$$\Delta x(n) = \sum_{i=1}^{\ell} p_i(n) x(\tau_i(n)) + \sum_{j=1}^{m} q_j(n) x(\sigma_i(n)), \quad n \geq n_0,$$

where:

- $\tau_i(n)$ is the delay term
  - $\tau_i(n) > 0$
  - $\tau_i(n) \leq n - 1$
  - $\lim_{n \longrightarrow +\infty} \tau_i(n) = +\infty$
- $p_i(n)$ and $q_j(n)$ are real functions.

- $\sigma_j(n)$ is the advance term
  - $\sigma_j(n) \geq n + 1$

The results was developed with Nedjem E. Ramdani and Ali Fuat yenicerioglu:

International Journal of Dynamical Systems and Differential Equations

Introduction

The second part of this work is dedicated at the oscillatory behavior of the mixed type differential equation with variable coefficients

$$x'(t) = \sum_{i=1}^{\ell} p_i(t)x(\tau_i(t)) + \sum_{j=1}^{m} q_j(t)x(\sigma_i(t)), \quad t \geq t_0$$

where:

- $\tau_i(t)$ is the delay term
  - $\tau_i(t) > 0$
  - $\tau_i(t) < t$
  - $\lim\limits_{t \longrightarrow +\infty} \tau_i(t) = +\infty$
- $p_i(t)$ and $q_j(t)$ are real functions
- $\sigma_j(t)$ is the advance term
  - $\sigma_j(t) > t$

Difference equations

Introduction

The equation

$$\Delta x(n) = \sum_{i=1}^{\ell} p_i x(n-i) + \sum_{j=1}^{m} q_j x(n+j)$$

was intoduced in *J. M. Ferreira and S. Pinelas,* ***Oscillatory mixed difference systems.*** *Adv. Differ. Equ. 2006, 1–18 (2006)*, where it has established the oscillatory criteria for the oscillatory behaviour of such equation.

This work is concerned with the behavior of the solutions of autonomous linear mixed type difference equations and the results will be obtained via an appropriate positive real root of the corresponding characteristic equation.

Introduction

The equation

$$\Delta x(n) = \sum_{i=1}^{\ell} p_i(n)x(n-\tau_i) + \sum_{j=1}^{m} q_j(n)x(n+\sigma_j), \quad n \geq 0$$

has been adequately introduced in

↳L. Berezansky and S. Pinelas, **Oscillation Properties for a Scalar Linear Difference Equation of Mixed Type**, Math. Bohemica, (2016); 141(2): 169-182.

↳S. Pinelas, **Asymptotic Behavior of a Scalar Linear Difference Equation of Mixed Equations**, UPI Journal of Math. and Biostatistics, (2018); 1(1): 13-21.

Introduction

The equation

$$\Delta x(n) = \sum_{i=1}^{\ell} p_i(n) x(n - \tau_i) \qquad n \geq 0$$

has been studied in:

- R. D. Driver, G. Ladas and P. N. Vlahos; **Asymptotic Behavior of a Linear Delay Difference Equation**, Proceedings of the American Mathematical Society, Vol. 115, No. 1, (1992), 105-112
- M. Pituk; **The limits of the solutions of a nonautonomous linear delay difference equation**, Comput. Math. Appl. 42 (2001), 543-550

Introduction

The equation

$$\Delta x(n) = \sum_{i=1}^{\ell} p_i(n)x(n-\tau_i) \qquad n \geq 0$$

has been studied in:

- Ch. G. Philos and I. K. Purnaras; **An asymptotic result and a stability criterion for linear nonautonomous delay difference equations**, Arch. Math. (Basel) 83 (2004), 243-255
- Ch. G. Philos and I. K. Purnaras; **Asymptotic behavior and stability to linear nonautonomous neutral delay difference equations**, J. Differ. Equations Appl. 11 (2005), 503-513

Introduction

A particular case of the equation

$$\Delta x(n) = \sum_{i=1}^{\ell} p_i(n) x(n-\tau_i) + \sum_{j=1}^{m} q_j(n) x(n+\sigma_j), \quad n \geq 0$$

is the linear autonomous mixed type difference equation

$$\Delta x(n) = \sum_{i=1}^{\ell} p_i x(n-i) + \sum_{j=1}^{m} q_j x(n+j)$$

➯J. M. Ferreira and S. Pinelas; **Oscillatory mixed difference systems**, Advances in Difference Equations, (2006) 1-18.

➯A. F. Yenicerioglu, S. Pinelas, and Y. Yan; **On the behavior of the solutions for linear autonomous mixed type dierence equation**, *Rend. Circ. Mat. Palermo II., Ser* **69**, 787–801 (2020).

Introduction

A solution of the equation

$$\Delta x(n) = \sum_{i=1}^{\ell} p_i(n) x(n - \tau_i) + \sum_{j=1}^{m} q_j(n) x(n + \sigma_j), \quad n \geq 0$$

is said to be **nonoscillatory** if it is either eventually positive or eventually negative.

Otherwise it is **oscillatory**.

An equation is called oscillatory if all its solutions are oscillatory.

Difference Equation

Theorem 1

Let $p_i(n)$ and $q_j(n)$ non negative sequences. If

$$\sigma(n) = \min_{1\le j\le m} \sigma_i(n)$$

$$Q(n) = \sum_{j=1}^{m} q_j(n)$$

$$\limsup_{n\to\infty} \sum_{k=\tau(n)}^{\sigma(n)-1} Q(k) > 1,$$

$$\tau(n) = \min_{1\le i\le l} \tau_i(n)$$

then

$$\Delta x(n) = \sum_{i=1}^{\ell} p_i(n)x(\tau_i(n)) + \sum_{j=1}^{m} q_j(n)x(\sigma_i(n))$$

is oscillatory.

Difference Equation

Proof

$x(n)$ is nonoscillatory $\Longrightarrow$ $$\Delta x(n) = \sum_{i=1}^{\ell} p_i(n)x(\tau_i(n)) + \sum_{j=1}^{m} q_j(n)x(\sigma_j(n)) \geq 0$$

$\Longrightarrow$ $x(n)$ is increasing

$$\sum_{k=\tau(n)}^{\sigma(n)-1} (x(n+1) - x(n)) = \sum_{k=\tau(n)}^{\sigma(n)-1} \sum_{i=1}^{\ell} p_i(k)x(\tau_i(k)) + \sum_{k=\tau(n)}^{\sigma(n)-1} \sum_{j=1}^{m} q_j(k)x(\sigma_j(k))$$

$$\left(\sum_{k=\tau(n)}^{\sigma(n)-1} P(k) + 1\right) x(\tau(n)) + \left(\sum_{k=\tau(n)}^{\sigma(n)-1} Q(k) - 1\right) x(\sigma(n)) \leq 0$$

Contadition!

Difference Equation

Corollary

Let $p_i(n)$ and $q_j(n)$ non negative sequences. If

$$\limsup_{n\to\infty} \sum_{j=1}^{m} \sum_{k=n-\rho}^{n+\rho-1} q_j(k) > 1$$

then

$$\Delta x(n) = \sum_{i=1}^{\ell} p_i(n) x(n-\tau_i) + \sum_{j=1}^{m} q_j(n) x(n+\sigma_j).$$

is oscillatory.

Difference Equation

Example

$$c_1 = \frac{e}{\ln 4}$$

$$\Delta x(n) = x(\underbrace{n-1}_{\tau_1(n)}) + x(\underbrace{n-2}_{\tau_2(n)}) + \frac{c_1}{3\ln(n+2)^{n+2}}\, x(\underbrace{n^2+1}_{\sigma_1(n)}) + \frac{2c_1}{3\ln(n+2)^{n+2}}\, x(\underbrace{n^2+2}_{\sigma_2(n)})$$

$$\tau(n) = \min_{1\le i\le 2} \tau_i(n) = n-2$$

$$\sigma(n) = \min_{1\le j\le 2} \sigma_j(n) = n^2+1$$

$$\sum_{k=n-1}^{n^2} Q(k) = \sum_{k=n-1}^{n^2} \left(\frac{c_1}{\ln(k+2)^{k+2}} \right)$$

$$\limsup_{n\to\infty} \sum_{k=\tau(n)}^{\sigma(n)-1} Q(k) > 1 \ ?$$

Difference Equation

Example (cont.)

$$\sum_{k=n-1}^{n^2} Q(k) = \sum_{k=n-1}^{n^2} \left(\frac{c_1}{\ln(k+2)^{k+2}} \right) > 1 \ ?$$

$$\sum_{k=n-1}^{n^2} \left(\frac{c_1}{\ln(k+2)^{k+2}} \right) \geq c_1 \sum_{k=n-1}^{n^2} \int_k^{k+1} \frac{ds}{\ln(s+2)^{(s+2)}} = c_1 \ln\left(\frac{\ln(n^2+3)}{\ln(n+1)} \right)$$

$$\int_{b-1}^{b} f(x)dx \geq f(b) \geq \int_b^{b+1} f(x)dx$$

$$\sum_{k=n-1}^{n^2} \left(\frac{c_1}{\ln(k+2)^{k+2}} \right) \leq c_1 \sum_{k=n-1}^{n^2} \int_{k-1}^{k} \frac{ds}{\ln(s+2)^{(s+2)}} = c_1 \ln\left(\frac{\ln(n^2+2)}{\ln(n)} \right)$$

Difference Equation

Example (cont.)

$$\lim_{n\to\infty} c_1 \ln\left(\frac{\ln(n^2+2)}{\ln(n)}\right) = \lim_{n\to\infty} c_1 \ln\left(\frac{\ln(n^2+3)}{\ln(n+1)}\right) = c_1 \ln(2) = \frac{e}{2} > 1$$

$$\Longrightarrow \quad \limsup_{n\to\infty} \sum_{k=n-1}^{n^2} Q(k) = \frac{e}{2} > 1$$

$\Longrightarrow$ **By Theorem 1, all solutions of equation oscillate!**

Difference Equation

Theorem 2

$$Q(n) = \sum_{j=1}^{m} q_j(n)$$

Let $p_i(n)$ and $q_j(n)$ non negative sequences. If

$$\sigma(n) = \min_{1\le j\le m} \sigma_i(n)$$

$$\limsup_{n\to\infty} \sum_{k=n}^{\sigma_j(n)-1} Q(k) > \frac{1}{e}$$

$$\tau(n) = \min_{1\le i\le l} \tau_i(n)$$

then

$$\Delta x(n) = \sum_{i=1}^{\ell} p_i(n)x(\tau_i(n)) + \sum_{j=1}^{m} q_j(n)x(\sigma_i(n))$$

is oscillatory.

Difference Equation

Proof

$$c_j(n) = \left(\frac{\sigma_j(n) - n}{1 + \sigma_j(n) - n}\right)^{1+\sigma_j(n)-n} \Longrightarrow \frac{1}{4} \le c_j(n) \le \frac{1}{e}$$

$$\limsup_{n\to\infty} \sum_{k=n}^{\sigma_j(n)-1} Q(k) > \frac{1}{e} \Longrightarrow \sum_{k=n}^{\sigma_j(n)-1} Q(k) > \frac{1}{e} + \epsilon_0$$

$$\left.\begin{array}{l} \frac{1}{4} \le c_j(n) \le \frac{1}{e} \\ \sum_{k=n}^{\sigma_j(n)-1} Q(k) > \frac{1}{e} + \epsilon_0 \end{array}\right\} \sum_{k=n}^{\sigma_j(n)-1} \frac{Q(k)}{c_j(n)} > d > 1$$

$x(n)$ is increasing $\Longrightarrow \dfrac{x(\tau_i(n))}{x(n+1)} \le 1, \quad \dfrac{x(\sigma_j(n))}{x(n+1)} \ge 1$

$$\Longrightarrow \frac{x(n)}{x(n+1)} = 1 - \sum_{i=1}^{l} p_i(n) \frac{x(\tau_i(n))}{x(n+1)} - \sum_{j=1}^{m} q_j(n) \frac{x(\sigma_j(n))}{x(n+1)}$$

Difference Equation

Proof (cont)

$$\frac{x(n)}{x(\sigma_j(n))} = \prod_{k=n}^{\sigma_j(n)-1} \frac{x(k)}{x(k+1)} \Longrightarrow \frac{x(n)}{x(\sigma_j(n))} \leq \prod_{k=n}^{\sigma_j(n)-1} (1 - Q(k))$$

Inequality of arithmetic and geometric means

$$\Longrightarrow \frac{x(\sigma_j(n))}{x(n)} \geq \left[1 - \frac{1}{\sigma_j(n) - n} \sum_{k=n}^{\sigma_j(n)-1} Q(k)\right]^{-(\sigma_j(n)-n)}$$

$$y(1-y)^\rho \leq \frac{\rho^\rho}{(1+\rho)^{1+\rho}}, \quad \forall y \in (0,1), \ \rho \in \mathbb{N}$$

$$\frac{x(\sigma_j(n))}{x(n)} \geq \sum_{k=n}^{\sigma_j(n)-1} Q(k) \left(\frac{1 + \sigma_j(n) - n}{\sigma_j - n}\right)^{1+\sigma_j(n)-n}$$

Difference Equation

Proof (cont)

$$\frac{x(\sigma_j(n))}{x(n)} \geq \sum_{k=n}^{\sigma_j(n)-1} \frac{Q(k)}{c_j(n)} \geq d \quad \Longrightarrow \quad \frac{x(\sigma_j(n))}{x(n)} > d^t$$

On the other hand…

$$0 < \frac{x(n)}{x(n+1)} = 1 - \sum_{i=1}^{l} p_i(n)\frac{x(\tau_i(n))}{x(n+1)} - \sum_{j=1}^{m} q_j(n)\frac{x(\sigma_j(n))}{x(n+1)}$$

$$\frac{x(\sigma(n))}{x(n)} Q(n) < 1 \quad \Longleftarrow \quad 0 < 1 - \sum_{j=1}^{m} q_j(n)\frac{x(\sigma(n))}{x(n)}$$

Difference Equation

Proof (cont)

$$\lim \theta(n) = +\infty \quad \Longrightarrow \quad \frac{x(\sigma(\theta(n)))}{x(\theta(n))} < \frac{1}{Q(\theta(n))} \leq \frac{1}{C} < +\infty$$

$$\Longrightarrow \quad \lim_{n\to\infty} \frac{x(\sigma(n))}{x(n)} \text{ exists}$$

Contradition

$$\frac{x(\sigma_j(n))}{x(n)} > d^t$$

Difference Equation

Corollary

Let $p_i(n)$ and $q_j(n)$ non negative sequences. If

$$\liminf_{n\to\infty} \sum_{j=1}^{m} \sum_{k=n-\tau_i}^{n+\sigma_i-1} q_j(k) > \frac{1}{e}$$

then

$$\Delta x(n) = \sum_{i=1}^{\ell} p_i(n) x(n-\tau_i) + \sum_{j=1}^{m} q_j(n) x(n+\sigma_j).$$

is oscillatory

Difference Equation

Example

$$\Delta x(n) = \frac{1}{3}x(\underbrace{n-1}_{\tau_1(n)}) + \frac{2}{3}x(\underbrace{n-4}_{\tau_2(n)}) + \frac{1}{3e}x(\underbrace{n+1}_{\sigma_1(n)}) + \frac{4}{5e}x(\underbrace{n+3}_{\sigma_2(n)})$$

$$\sigma_1(n) = n+1, \; \sigma_2(n) = n+3$$

$$\liminf_{n\to\infty} \sum_{k=n}^{n} Q(k) = \frac{17}{15e} > \frac{1}{e}$$

$$\liminf_{n\to\infty} \sum_{k=n}^{n+2} Q(k) = \frac{17}{5e} > \frac{1}{e}$$

$$\limsup_{n\to\infty} \sum_{k=n}^{\sigma_j(n)-1} Q(k) > \frac{1}{e}?$$

By Theorem 2, all solutions of equation oscillate!

Difference Equation

Theorem 3

Let $p_i(n)$ and $q_j(n)$ non positive sequences. If

$$\sigma(n) = \min_{1\le j\le m} \sigma_i(n)$$

$$P(n) = \sum_{i=1}^{\ell} p_i(n)$$

$$\limsup_{n\to\infty} \sum_{k=\tau(n)}^{\sigma(n)-1} P(k) < -1.$$

$$\tau(n) = \min_{1\le i\le l} \tau_i(n)$$

then

$$\Delta x(n) = \sum_{i=1}^{\ell} p_i(n) x(\tau_i(n)) + \sum_{j=1}^{m} q_j(n) x(\sigma_i(n))$$

is oscillatory.

Difference Equation

Corollary

Let $p_i(n)$ and $q_j(n)$ non positive sequences. If

$$\limsup_{n\to\infty} \sum_{i=1}^{\ell} \sum_{k=n-\rho}^{n+\rho-1} p_i(k) < -1$$

then

$$\Delta x(n) = \sum_{i=1}^{\ell} p_i(n) x(n-\tau_i) + \sum_{j=1}^{m} q_j(n) x(n+\sigma_j).$$

is oscillatory.

Difference Equation

Example

$$\frac{3}{2\ln 2}$$

$$\Delta x(n) = -\frac{c_2}{3n}x(\underbrace{[0.5n]}_{\tau_1(n)}) - \frac{2c_2}{3n}x(\underbrace{[n^{0.5}]}_{\tau_2(n)}) - x(\underbrace{n+1}_{\sigma_1(n)}) - 2x(\underbrace{n^2+2}_{\sigma_2(n)})$$

$$\liminf_{n\to\infty} \sum_{k=\tau(n)}^{\sigma(n)-1} P(k) < -1\ ?$$

$\Rightarrow$ $\tau(n) = [0.5n]$

$\Rightarrow$ $\sigma(n) = n+1$

$$\Rightarrow \sum_{k=[0.5n]}^{n} P(k) = \sum_{k=[0.5n]}^{n}\left(\frac{-c_2}{3k} + \frac{-2c_2}{3k}\right) = \sum_{k=[0.5n]}^{n}\left(\frac{-c_2}{k}\right)$$

Difference Equation

Example (cont.)

$$\sum_{k=[0.5n]}^{n} \left(\frac{-c_2}{k}\right) < -1. \quad ?$$

$$-c_2 \sum_{k=[0.5n]}^{n} \int_{k-1}^{k} \frac{ds}{s} \le -\sum_{k=[0.5n]}^{n} \frac{c_2}{k} \le -c_2 \sum_{k=[0.5n]}^{n} \int_{k}^{k+1} \frac{ds}{s}$$

$$\lim_{n\to\infty} \left[-\sum_{k=[0.5n]}^{n} \frac{c_2}{k}\right] = -c_2 \ln 2 = -\frac{3}{2} < -1$$

By Theorem 3, all solutions of equation oscillate!

Difference Equation

Theorem 4

Let $p_i(n)$ and $q_j(n)$ non positive sequences. If

$$\sigma(n) = \min_{1\le j\le m} \sigma_i(n)$$

$$P(n) = \sum_{i=1}^{\ell} p_i(n)$$

$$\liminf_{n\to\infty} \sum_{k=\tau_i(n)}^{n-1} P(k) < -\frac{1}{e}$$

$$\tau(n) = \min_{1\le i\le l} \tau_i(n)$$

then

$$\Delta x(n) = \sum_{i=1}^{\ell} p_i(n)x(\tau_i(n)) + \sum_{j=1}^{m} q_j(n)x(\sigma_i(n))$$

is oscillatory.

Difference Equation

Corollary

Let $p_i(n)$ and $q_j(n)$ non positive sequences. If

$$\liminf_{n\to\infty} \sum_{i=1}^{\ell} \sum_{k=n-\tau_i}^{n-1} p_i(k) < -\frac{1}{e}$$

then

$$\Delta x(n) = \sum_{i=1}^{\ell} p_i(n) x(n-\tau_i) + \sum_{j=1}^{m} q_j(n) x(n+\sigma_j).$$

is oscillatory.

Difference Equation

Example

$$\Delta x(n) = -\frac{1}{3e}x\underbrace{(n-1)}_{\tau_1(n)} - \frac{5}{6e}x\underbrace{(n-2)}_{\tau_2(n)} - \frac{1}{2}x\underbrace{(n+1)}_{\sigma_1(n)} - \frac{1}{2}x\underbrace{(n+5)}_{\sigma_2(n)}$$

$$\tau_1(n) = n-1,\ \tau_2(n) = n-2$$

$$\liminf_{n\to\infty} \sum_{k=n-1}^{n-1} P(k) = -\frac{7}{6e} < -\frac{1}{e}$$

$$\liminf_{n\to\infty} \sum_{k=n-2}^{n-1} P(k) = -\frac{7}{3e} < -\frac{1}{e}$$

By Theorem 4, all solutions of equation oscillate!

$$\liminf_{n\to\infty} \sum_{k=\tau_i(n)}^{n-1} P(k) < -\frac{1}{e}?$$

Difference Equation

Lets now to considerer a particular case of the difference equation with constant coefficients:

$$\Delta x(n) = \sum_{i=1}^{\ell} p_i x(n-i) + \sum_{j=1}^{m} q_j x(n+j)$$

where:

- p_i and q_j are real numbers,
- ℓ and m are positive integers
- $\Delta x(n) = x(n+1) - x(n)$ is the forward operator

Difference Equation

The **characteristic equation** is given by

$$1 = \lambda - \sum_{i=1}^{\ell} p_i \lambda^{-i} - \sum_{j=1}^{m} q_j \lambda^j$$

We say that the characteristic equation has the ***Property A*** if

$$\mu(\lambda_0) = \frac{1}{\lambda_0} \left\{ \sum_{i=1}^{\ell} i \left| p_i \right| \lambda_0^{-i} + \sum_{j=1}^{m} j \left| q_j \right| \lambda_0^{j} \right\} < 1$$

Difference Equation

Theorem 1: Let λ_0 *be a positive real root of the characteristic equation with the Property A.*

$$\beta(\lambda_0) = \frac{1}{\lambda_0}\left(\sum_{i=1}^{\ell} i p_i \lambda_0^{-i} - \sum_{j=1}^{m} j q_j \lambda_0^{j}\right)$$

Then, for any ϕ

$$\lim_{n\to\infty}\left[\lambda_0^{-n} x(n)\right] = \frac{L(\lambda_0;\phi)}{1+\beta(\lambda_0)}$$

$$L(\lambda_0;\phi) = \phi(0) + \frac{1}{\lambda_0}\left\{\sum_{i=1}^{\ell} p_i \lambda_0^{-i}\left(\sum_{s=-i}^{-1}\phi(s)\lambda_0^{-s}\right) - \sum_{j=1}^{m} q_j \lambda_0^{j}\left(\sum_{s=0}^{j-1}\phi(s)\lambda_0^{-s}\right)\right\}.$$

Difference Equation

Theorem 2: Let λ_0 *be a positive real root of the characteristic equation with the Property A.*

$$\beta(\lambda_0) = \frac{1}{\lambda_0}\left(\sum_{i=1}^{\ell} i p_i \lambda_0^{-i} - \sum_{j=1}^{m} j q_j \lambda_0^{j}\right)$$

Then, for any ϕ

$$|x(n)| \le N(\lambda_0) \parallel \phi \parallel \lambda_0^{n}$$

Moreover, *the solution of the mixed type difference equation*

$$\Delta x(n) = \sum_{i=1}^{\ell} p_i x(n-i) + \sum_{j=1}^{m} q_j x(n+j)$$

$$N(\lambda_0) = \frac{1+\mu(\lambda_0)}{1+\beta(\lambda_0)} + \mu(\lambda_0) k(\lambda_0)\left(1 + \frac{1+\mu(\lambda_0)}{1+\beta(\lambda_0)}\right)$$

is:

i. *uniformly stable if* $\lambda_0 = 1$

ii. *uniformly asymptotically stable if* $\lambda_0 < 1$

iii. *unstable* $\lambda_0 > 1$

$$k(\lambda_0) = max\{\lambda_0^{\ell}, \lambda_0^{-m}\}$$

Difference Equation

Lemma: Let $r = \max\{\ell, m\}$.

$$\Longrightarrow \quad \frac{1}{r+1} + \sum_{i=1}^{\ell} p_i \left(\frac{r+1}{r}\right)^i + \sum_{j=1}^{m} q_j \left(\frac{r}{r+1}\right)^j > 0$$

$$\Longrightarrow \quad \frac{1}{r} - \sum_{i=1}^{\ell} p_i \left(\frac{r}{r+1}\right)^i - \sum_{j=1}^{m} q_j \left(\frac{r+1}{r}\right)^j > 0$$

$$\Longrightarrow \quad \sum_{i=1}^{\ell} i|p_i| \left(\frac{r+1}{r}\right)^{i+1} + \sum_{j=1}^{m} j|q_j| \left(\frac{r+1}{r}\right)^{j-1} \leq 1$$

Then, in the interval $\left(\frac{r}{r+1}, \frac{r+1}{r}\right)$ the characteristic equation

$$1 = \lambda - \sum_{i=1}^{\ell} p_i \lambda^{-i} - \sum_{j=1}^{m} q_j \lambda^j$$

has a unique positive root λ_0 , and this root satisfies the *Property A*.

Difference Equation

Proof: Define
$$F(\lambda) = \lambda - 1 - \sum_{i=1}^{\ell} p_i \lambda^{-i} - \sum_{j=1}^{m} q_j \lambda^j \qquad \lambda \in \left[\frac{r}{r+1}, \frac{r+1}{r}\right]$$

$$F\left(\frac{r}{r+1}\right) = -\frac{1}{r+1} - \sum_{i=1}^{\ell} p_i \left(\frac{r+1}{r}\right)^i - \sum_{j=1}^{m} q_j \left(\frac{r}{r+1}\right)^j$$

$$\frac{1}{r+1} + \sum_{i=1}^{\ell} p_i \left(\frac{r+1}{r}\right)^i + \sum_{j=1}^{m} q_j \left(\frac{r}{r+1}\right)^j > 0$$

$$F\left(\frac{r+1}{r}\right) = \frac{1}{r} - \sum_{i=1}^{\ell} p_i \left(\frac{r}{r+1}\right)^i - \sum_{j=1}^{m} q_j \left(\frac{r+1}{r}\right)^j$$

$$\frac{1}{r} - \sum_{i=1}^{\ell} p_i \left(\frac{r}{r+1}\right)^i - \sum_{i=1}^{m} q_j \left(\frac{r+1}{r}\right)^j > 0$$

Difference Equation

Proof:

$$F'(\lambda) = 1 + \sum_{i=1}^{\ell} i p_i \lambda^{-i-1} - \sum_{j=1}^{m} j q_j \lambda^{j-1}$$

$$\geq 1 - \sum_{i=1}^{\ell} i|p_i|\lambda^{-i-1} - \sum_{j=1}^{m} j|q_j|\lambda^{j-1}$$

$$> 1 - \sum_{i=1}^{\ell} i|p_i|\left(\frac{r}{r+1}\right)^{-i-1} - \sum_{j=1}^{m} j|q_j|\left(\frac{r+1}{r}\right)^{j-1} \geq 0$$

$$\sum_{i=1}^{\ell} i|p_i|\left(\frac{r+1}{r}\right)^{i+1} + \sum_{j=1}^{m} j|q_j|\left(\frac{r+1}{r}\right)^{j-1} \leq 1$$

Then F is incresing

$F(\lambda) = 0$ has an unique solution in the interval $\left(\frac{r}{r+1}, \frac{r+1}{r}\right)$

Difference Equation

Proof:

$$\frac{1}{\lambda_0}\left\{\sum_{i=1}^{\ell} i\,|p_i|\,\lambda_0^{-i} + \sum_{j=1}^{m} j\,|q_j|\,\lambda_0^{j}\right\}$$

$$= \sum_{i=1}^{\ell} i\,|p_i|\,\lambda_0^{-i-1} + \sum_{j=1}^{m} j\,|q_j|\,\lambda_0^{j-1}$$

$$< \sum_{i=1}^{\ell} i|p_i|\left(\frac{r}{r+1}\right)^{-i-1} + \sum_{j=1}^{m} j|q_j|\left(\frac{r+1}{r}\right)^{j-1} \leq 1.$$

$$\sum_{i=1}^{\ell} i|p_i|\left(\frac{r+1}{r}\right)^{i+1} + \sum_{j=1}^{m} j|q_j|\left(\frac{r+1}{r}\right)^{j-1} \leq 1$$

Difference Equation

Corollary: Let $r = \max\{\ell, m\}$.

$$\Longrightarrow \quad \frac{1}{r+1} + \sum_{i=1}^{\ell} p_i \left(\frac{r+1}{r}\right)^i + \sum_{j=1}^{m} q_j \left(\frac{r}{r+1}\right)^j > 0$$

$$\Longrightarrow \quad \frac{1}{r} - \sum_{i=1}^{\ell} p_i \left(\frac{r}{r+1}\right)^i - \sum_{j=1}^{m} q_j \left(\frac{r+1}{r}\right)^j > 0$$

$$\Longrightarrow \quad \sum_{i=1}^{\ell} i|p_i| \left(\frac{r+1}{r}\right)^{i+1} + \sum_{j=1}^{m} j|q_j| \left(\frac{r+1}{r_m}\right)^{j-1} \leq 1$$

Then the solution of the equation $\Delta x(n) = \sum_{i=1}^{\ell} p_i x(n-i) + \sum_{j=1}^{m} q_j x(n+j)$

is:

- asymptotically stable if $\sum_{i=1}^{\ell} p_i + \sum_{j=1}^{m} q_j < 0$
- unstable if $\sum_{i=1}^{\ell} p_i + \sum_{i=1}^{m} q_j > 0$

Difference Equation

Example 1:

$$\Delta x(n) = \sum_{i=1}^{2} \left(-\frac{1}{4}\right)^{i} x(n-i) + \sum_{j=1}^{2} \left(-\frac{1}{4}\right)^{j+1} x(n+j)$$

$$1 = \lambda - \sum_{i=1}^{2} \left(-\frac{1}{4}\right)^{i} \lambda^{-i} - \sum_{j=1}^{2} \left(-\frac{1}{4}\right)^{j+1} \lambda^{j}$$

$$\lambda^4 + 12\lambda^3 - 16\lambda^2 + 4\lambda - 1 = 0$$

$$\lambda_1 = 1, \ \lambda_2 = -13.2324, \ \lambda_3 = 0.1162 + 0.2491i, \ \lambda_4 = 0.1162 - 0.2491i$$

By the **Theorem 2**, the solution is uniformly stable

Difference Equation

Example 2:

$$\Delta x(n) = -\frac{1}{9}x(n-1) - \frac{1}{4}x(n+1)$$

$$45\lambda^2 - 36\lambda + 4 = 0$$

$\lambda_1 = \frac{2}{15}$ → Don't verify the *Property A*

$\lambda_2 = \frac{2}{3}$ → Verify the *Property A*

→ By the **Theorem 2**, the solution uniformly asymptotically stable

→ By the **Corollary** → $r = 1$ and $\sum_{i=1}^{\ell} p_i + \sum_{j=1}^{m} q_j = -\frac{1}{9} - \frac{1}{4} < 0$

→ The solution uniformly asymptotically stable

Differential equations

Introduction

Recalling the mixed type differential equation with variable coefficients

$$x'(t) = \sum_{i=1}^{\ell} p_i(t)x(\tau_i(t)) + \sum_{j=1}^{m} q_j(t)x(\sigma_i(t)), \quad t \geq t_0$$

where:

- $\tau_i(t)$ is the delay term
  - $\tau_i(t) > 0$
  - $\tau_i(t) < t$
  - $\lim_{t \longrightarrow +\infty} \tau_i(t) = +\infty$
- $\sigma_j(t)$ is the advance term
  - $\sigma_j(t) > t$
- $p_i(t)$ and $q_j(t)$ are real functions

Differential Equation

Theorem 1

Let $p_i(\cdot)$ and $q_j(\cdot)$ non negative sequences. If

$$\limsup_{n\to\infty} \sum_{k=\tau(n)}^{\sigma(n)-1} Q(\limsup_{t\to\infty} \int_{\tau(t)}^{\sigma(t)} Q(s)ds > 1$$

then

$$x'(t) = \sum_{i=1}^{\ell} p_i(t)x(\tau_i(t)) + \sum_{j=1}^{m} q_j(t)x(\sigma_i(t))_{n}))$$

is oscillatory.

Differential Equation

Theorem 3

Let $p_i(\cdot)$ and $q_j(\cdot)$ non positive sequences. If

$$\limsup_{n\to\infty} \sum_{k=\tau(n)}^{\sigma(n)-1} P(k) \limsup_{t\to+\infty} \int_{\tau(t)}^{\sigma(t)} P(s)ds < -1$$

then

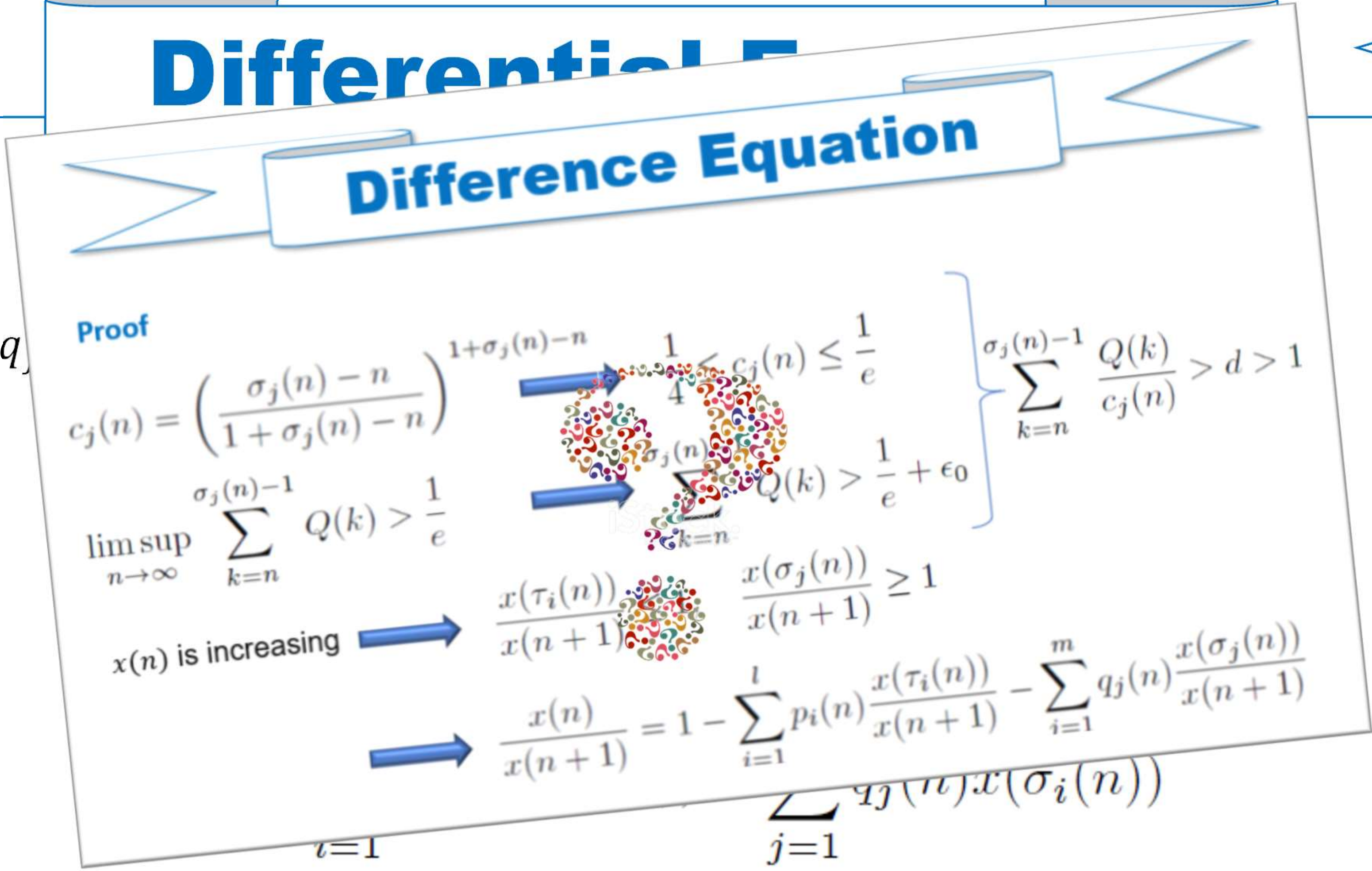
$$\Delta x'(t) = \sum_{i=1}^{\ell} p_i(t)x(\tau_i(t)) + \sum_{j=1}^{m} q_j(t)x(\sigma_i(t)))$$

is oscillatory.

Differential E

Theorem 2

Let $p_i(\cdot)$ and q_j



then

$$\sum_{i=1} \qquad \sum_{j=1} q_j(n) x(\sigma_i(n))$$

is oscillatory.

References

- T. Asada and H. Yoshida, **Stability, instability and complex behavior in macrodynamic models with policy lag**. Discrete Dyn. Nat. Soc. 5 (2001), 281–295.
- D. Dubois and K. E. Stecke, **Dynamic analysis of repetitive decision-free discrete-event processes: applications to production systems**. Ann. Oper. Res. 26 (1990), 323–347.
- R. Frisch and H. Holme, **The characteristic solutions of a mixed difference and differential equation occurring in economic dynamics**. Econometrica 3 (1935), 225–239.
- G. Gandolfo, **Economic Dynamics**. Springer, Berlin, 2010

References

- S. Elaydi, **An Introduction to Difference Equations**, 3rd edn. Springer, New York (2005)
- R. P. Agarwal, **Difference Equations and Inequalities**, **Theory**, **Methods and Applications**, 2nd edn. Marcel Dekker, New York (2000)
- I. Gyori and G. Ladas, **Oscillation Theory of Delay Differential Equations**, Mathematical Monographs, Oxford University Press, New York (1991)
- V. Lakshmikantham, D. Trigiante, **Theory of Difference Equations: Numerical Methods and Applications**, Academic Press Inc., New York (1988)
- W. G. Kelly and A. C. Peterson, **Difference Equations, An Introduction with Applications**, Academic Press, New York (1991)

References

- K. Das, M. Naga Srinivas, N. Gazi and S. Pinelas, **Stability of the zero solution of nonlinear tumor growth cancer model under the influence of white noise**, International Journal of Systems Applications, Engineering & Development, Volume 12, 2018, 12-27

- J. Mallet-Paret**, The Fredholm alternative for functional differential equations of mixed type**, J. Dynamics Differential Equations 11 (1999), 1-47

- A. Rustichini**, Hopf bifurcation for functional differential equations of mixed type**, J. Dynamics Differential Equations, 1 (1989) 145-177

References

- L. Berenzansky and E. Braverman, **Some oscillation problems for a second order linear delay differential equations**, J. Math. Anal. Appl., 220 (1998), 719740.
- V. Iakoveleva and C. J. Vanegas, **On the oscillation of differential equations with delayed and advanced arguments**, Electronic Journal of Differential Equation, 13 (2005), 57-63.
- T. Krisztin, Nonoscillations for functional differential equations of mixed type, Journal of Mathematical Analysis and Applications 254 (2000) 326-345
- R. P. Agarwal, S.R. Grace and D. O'Reagan, **Oscillation Theory for Difference and Functional Differential Equations**. Kluwer, 2000
- R. W. James and M. H. Belz, **The significance of the characteristic solutions of mixed difference and differential equations**. Econometrica 6 (1938), 326–343