# Cours 7
# Mathematical modeling of renewable resource management


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# Introduction

## Why use wastewater treatment ?

- The Lack of access to safe drinking water for 40 % of world population : United-Nations.
- The impact of pollutant on receiving environments and ecosystems.
- climate change and water scarcity.

A. Lipponen et N. Bonvoisin,
***Enhancement of water resources 2016.*** Technical report.
United Nations Educational, Scientific and Cultural Organization, 2016.

## Wastewater treatment station

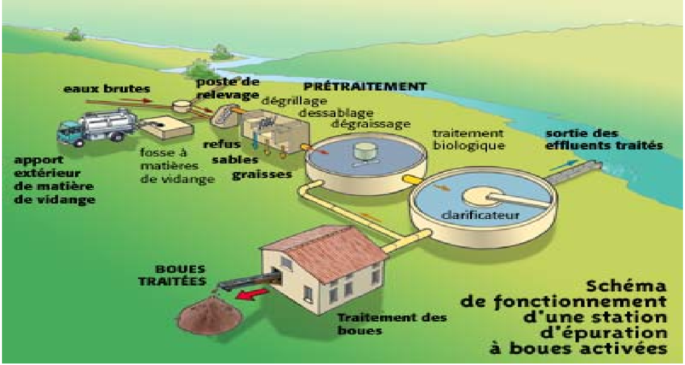



Figure – Activated sludge station diagram.

## Activated sludge process

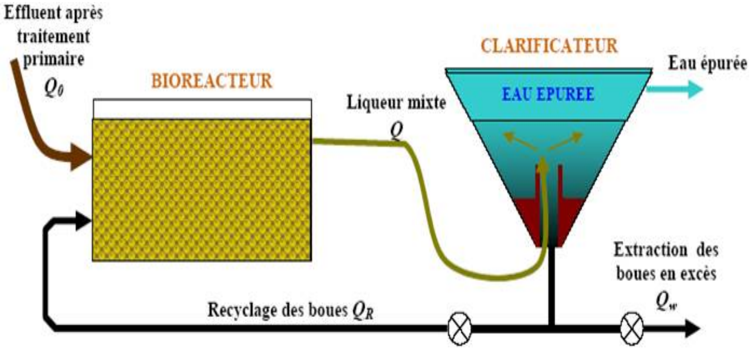



Figure – Activated sludge station diagram.

## Stability Concept

Consider a dynamical system

$$(DS)\begin{cases} \dot{x}(t) = f(x(t)) \ a.e. \ t \in [0, T) \\ x(0) = x_0. \end{cases} \tag{2.1}$$

and $\overline{x}$ an equilibrium point : $f(\overline{x}) = 0$.
We say that $\overline{x}$ is stable provided that for each $\varepsilon > 0$, there exists $\delta > 0$ s.t. if $\|x_0 - \overline{x}\| \leq \delta$ then $\|x(t) - \overline{x}\| \leq \varepsilon$.
We say that $\overline{x}$ is asymptotically stable if moreover

$$\lim_{t \to +\infty} \|x(t) - \overline{x}\| = 0.$$

The stability becomes global if it's fulfilled for any initial condition $x_0$.

# Anaerobic activated sludge process

Three phenomenas are considered :

- Substrate degradation ($S$).
- Bacteria growth in the aerator ($X$).
- The recycle of bacteria biomass from the settler ($X_r$).

Assumptions :

► All solid components will settle and concentrate at the bottom of the settling tank.

► The sedimentation of soluble organic matter is not significant.

## Mathematical model

The mass balance of the various constituents gives the following dynamical system model

$$(S)\quad \begin{cases} \dfrac{ds}{dt} = -\dfrac{\mu(s)x}{Y} - (1+r)Ds + Ds_{in} \ ; & s(0) = s_0 \\ \dfrac{dx}{dt} = \mu(s)x - (1+r)Dx + rDx_r \ ; & x(0) = x_0 \\ \dfrac{dx_r}{dt} = \nu(1+r)Dx - \nu(w+r)Dx_r \ ; & x_r(0) = x_{r0} \end{cases}$$

Where
$w = \dfrac{Q_w}{Q_{in}}$, $\nu = \dfrac{V_a}{V_s}$, $D = \dfrac{Q_{in}}{V_a}$, $r = \dfrac{Q_r}{Q_{in}}$ and $\mu(s) := \dfrac{ms}{k+s}$.

## Equilibria

The system (S) always has a boundary equilibrium $P_0 = (\frac{s_{in}}{1+r}, 0, 0)$.

On the other hand,

If

$$(1+r)D\frac{w}{w+r} < \mu(\frac{s_{in}}{1+r}) \tag{2.2}$$

Then, there exists an interior equilibrium point $P_1 = (s^*, x^*, x_r^*)$ such that

$$x_r^* = \frac{1+r}{w+r}x^* \ , \ x^* = \frac{1}{Y}\frac{w+r}{r}[\frac{s_{in}}{1+r} - s^*] \ , \ s^* = \mu^{-1}((1+r)D\frac{w}{w+r})$$

## Stability of $P_0$

Examine firstly, the global stability of $P_0$, whenever the condition (2.2) is not fulfilled.

### Theorem

If condition 2.2 is not fulfilled then the equilibrium point $P_0$ is globally asymptotically stable.

The idea of proof is based on the following Lyapunov function :

$$V(s, x, x_r) = Y \int_{\bar{s}}^{s} (1 - \frac{\mu(\bar{s})}{\mu(\xi)}) d\xi + x + \frac{r}{\nu(w+r)} x_r$$

Where $\bar{s} := \dfrac{s_{in}}{1+r}$.

## Stability of $P_1$

Suppose now that the condition is fulfilled.

### Theorem

Under the condition

$$(1+r)D\frac{w}{w+r} < \mu(\frac{s_{in}}{1+r}), \tag{2.3}$$

$P_0$ is instable and $P_1$ is globally asymptotically stable.

The idea of the proof is based on the following Lyapunov function :

$$\begin{aligned} V(s,x,x_r) \quad &:= \quad Y\frac{w+r}{r}\int_{s^*}^{s}(1-\frac{\mu(s^*)}{\mu(\xi)})d\xi + \frac{w+r}{r}\int_{x^*}^{x}(1-\frac{x^*}{\xi})d\xi \\ &+ \quad \frac{1}{v}\int_{x_{r^*}}^{x_r}(1-\frac{x_{r^*}}{\xi})d\xi \end{aligned}$$

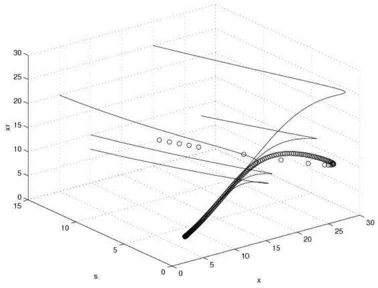



Figure – Global stability of interior equilibrium.

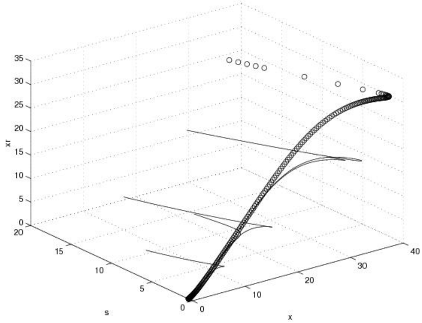



Figure – Global stability of boundary equilibrium.

# $\mu$ and $S_{in}$ not well known

What is about "stability theorem" whenever full information about growth function $\mu$ and $s_{in}$ is not available ?

Indeed, due to metabolic variations and the influence of many physic-chemical factors (PH, Temperature, oxygen, ...),

It's very hard to have an accurate idea of $\mu$.

We expect that there exist a domain of stability instead a unique stable equilibrium point.

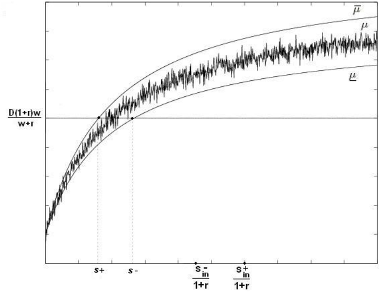



Figure – $\mu$ not well known and its bounds

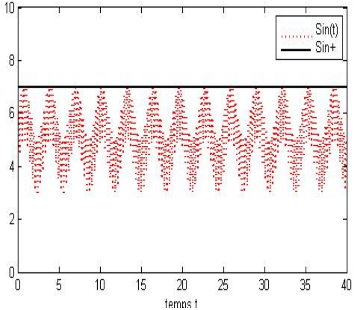



Figure – $S_{in}$ not well known and its bounds

# New assumptions

$H_1 -$ $D$, $r$, $w$ and $\nu$ are positive constants.

$H_2 -$ There exist two functions $\underline{\mu}$ and $\overline{\mu}$ satisfying the hypothesis $A_2$ such that

$$\underline{\mu}(s) \leq \mu(s) \leq \overline{\mu}(s), \quad \forall\, s \geq 0\,.$$

$H_3 -$ $s_{in}(t)$ is a not well known time varying function but bounded by :

$$s_{in}^{-} \leq s_{in}(t) \leq s_{in}^{+}, \quad \forall t \geq 0$$

where $s_{in}^{-}$ and $s_{in}^{+}$ are a given positive constants.

Consider now the two following systems

$$(S_+)\quad \begin{cases} \dot{s} = -\dfrac{\overline{\mu}(s)}{Y}x - (1+r)Ds + Ds_{in}^{+} \\ \dot{x} = \overline{\mu}(s)x - (1+r)Dx + rDx_r \\ \dot{x}_r = \nu(1+r)Dx - \nu(w+r)Dx_r \end{cases}$$

and

$$(S_-)\quad \begin{cases} \dot{s} = -\dfrac{\underline{\mu}(s)}{Y}x - (1+r)Ds + Ds_{in}^{-} \\ \dot{x} = \underline{\mu}(s)x - (1+r)Dx + rDx_r \\ \dot{x}_r = \nu(1+r)Dx - \nu(w+r)Dx_r \end{cases}$$

Systems $(S_+)$ and $(S_-)$ fulfill Assumptions $A_1$ and $A_2$, so,

- $(S_+)$ admits an unique global equilibrium point $P_+ = (s^+, x^+, x_r^+)$ and $(S_-)$ admits an unique global equilibrium point $P_- = (s^-, x^-, x_r^-)$.

Idea : prove that trajectories of system $S$ remain in the domain bounded by the equilibrium points $P_+ = (s^+, x^+, x_r^+)$ and $P_- = (s^-, x^-, x_r^-)$.

### Theorem (Serhani, Gouzé and Raissi)

Any trajectory $(s, x, x_r)$ of system $(S)$ starting in $\Omega$ converges towards the domain $U := \{(s, x, x_r) \in \Omega \ : \ s^- - \Delta \leq s \leq s^+ + \Delta \ , \ x^- \leq x \leq x^+ \ , \ x_r^- \leq x_r \leq x_r^+\}$, where $\Delta := \dfrac{x^+ - x^-}{Y}$.

The idea of proof is based on the variable change $z := x + Ys$ and properties of cooperative and monotone dynamical systems.

# Stability domain

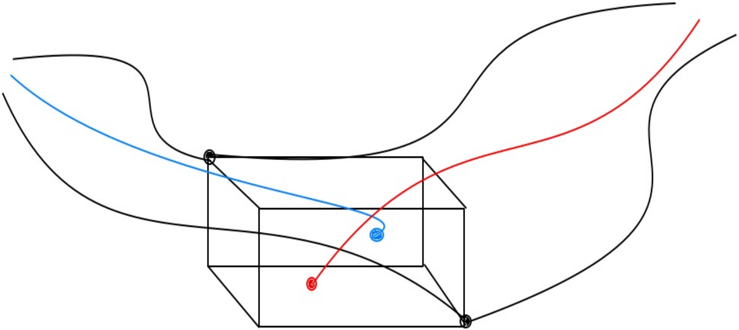


Figure – Stability domain.

## Problem

The reactions inside the bioreactor are often nonlinear, not well known and are subject external perturbations

Consequence :

- $S_{in}$ and $\mu$ are not well known and are subject disturbances.
- Idea : Invoke the stabilization concept.

## Stabilization Concept

Consider a controlled dynamical system

$$(DS)\begin{cases} \dot{x}(t) = f(x(t), u(t)) \ a.e. \ t \in [0, T) \\ x(0) = x_0. \end{cases} \tag{3.1}$$

We say that $(DS)$ is stabilizable if there exists a feedback control $u : \mathbb{R}^n \longrightarrow u(x) \in U$ such that the ODE

$$\dot{x}(t) = g(x(t))$$

is globally asymptotically stable (GAS).
where $g(x) := f(x, u(x))$.

# Stabilization

In case of discontinuous control, use the differential inclusion

$$(DS)\begin{cases} \dot{x}(t) \in F(x(t))\ a.e.\ t \in [0, T) \\ x(0) = x_0. \end{cases} \tag{3.2}$$

the set valued map $F$ must satisfy some assumptions (Multifunction of Marchaud) to guarantees the existence.

- Filippov set-valued map :
  $F_F(x) := \cap_{\lambda(N)=0} \cap_{\delta>0} \overline{co} f(x, u(x + \delta B \setminus N)), \forall x \in \mathbb{R}^n$
- Krasovskii set-valued map : $F_K(x) := \cap_{\delta>0} \overline{co} f(x, u(x + \delta B))$, $\forall x \in \mathbb{R}^n$

# Wastewater treatment station

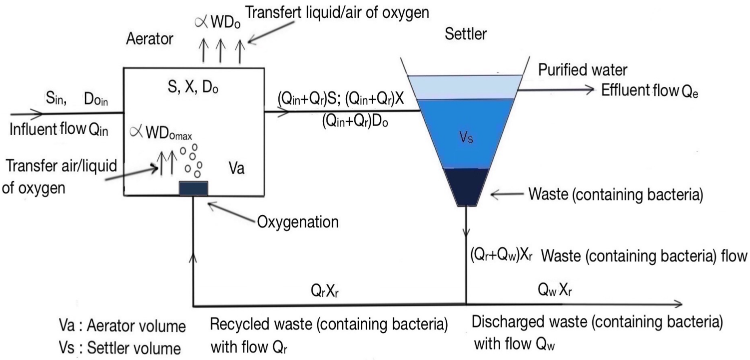
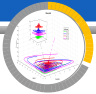



Figure – Activated sludge station diagram.

# Mathematical model

$$
(\mathcal{S}_2):\begin{cases}
\dfrac{dx(t)}{dt} = \mu(x(t), Do(t))x(t) - D(t)(1+r)x(t) + rD(t)x_r(t) \\[2ex]
\dfrac{dx_r(t)}{dt} = D(t)(1+r)x(t) - D(t)(\beta+r)x_r(t) \\[2ex]
\dfrac{dS(t)}{dt} = -\dfrac{\mu(x(t), Do(t))}{Y_f}x(t) - D(t)(1+r)S(t) + D(t)S_{in} \\[2ex]
\dfrac{dDo(t)}{dt} = -K_0\dfrac{\mu(x(t), Do(t))}{Y_f}x(t) - D(t)(1+r)Do(t) + D(t)Do_{in} + \\
\qquad \alpha W[Do_{max} - Do(t)]
\end{cases}
$$



## Hypotheses for activated sludge problem

The biological constraints may be expressed mathematically with the following assumptions :

- $S$ et $Do$ are the known (accuracy estimated) states .
- $x$ et $x_r$ are the unknown states.
- $S_{in}^{-}(t) \leq S_{in}(t) \leq S_{in}^{+}(t)$
- $Do_{in}^{-}(t) \leq Do_{in}(t) \leq Do_{in}^{+}(t)$
- $\underline{\mu(\xi)} \leq \mu^{-}(t,\xi) \leq \mu(t,\xi) \leq \mu^{+}(t,\xi) \leq \overline{\mu(\xi)}, \forall t \geq 0, \forall \xi \in \mathbb{R}^2$

# Consequence

- $x$, $x_r$ are difficult to estimate accurately.
- Very hard to build a feedback stabilizing control law with these states.

## Solution

- Construct observer intervals to missing states.
- Build a feedback control with these observers stabilizing the system around a suitable level.

# Observer Intervals

The model $(S_2)$ can be reformulated as :

$$(\mathcal{S}r_2) : \begin{cases} \dfrac{dX_1}{dt} = C_1 R(t) X_1(t) + A_{11}(t) X_1(t) \\[2ex] \dfrac{dX_2}{dt} = C_2 R(t) X_1(t) + A_{22}(t) X_2(t) + B_2(t) \\[2ex] Y = H X_2(t) \end{cases} \tag{3.3}$$

with

$X_1 = \begin{pmatrix} x_r \\ x \end{pmatrix}$, $X_2 = \begin{pmatrix} S \\ Do \end{pmatrix}$, $R(t) = \begin{pmatrix} 0 & \mu(t) \end{pmatrix}$, $H = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

and $B_2(t) = \begin{pmatrix} D(t) S_{in} \\ \\ D(t) Do_{in} + \alpha W Do_{max} \end{pmatrix}$

Idea : Use a formulation free of kinetic function.

- Define a new state $Z$ by

$$Z(t) = N \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix}.$$

where $N \in \mathcal{M}_{2,4}(\mathbb{R})$.

- Choose $N = \begin{pmatrix} N_1 & N_2 \end{pmatrix}$ : $\dfrac{dZ}{dt}$ becomes a function of $Z$ and $X_2$ :

$$\frac{dZ}{dt} = \beta_1 Z(t) + \beta_2 X_2(t) + N.B(t)$$

with $B(t) = \begin{pmatrix} 0 \\ B_2(t) \end{pmatrix}$ and $\beta_1, \beta_2$ depend on parameters and $N$.

Consider the systems

$$
\begin{aligned}
\frac{dZ^{+}}{dt} &= \beta_1 Z^{+}(t) + \beta_2 X_2(t) + N_2 B_2^{+} \\
Z^{+}(0) &= N\alpha^{+}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{dZ^{-}}{dt} &= \beta_1 Z^{-}(t) + \beta_2 X_2(t) + N_2 B_2^{-} \\
Z^{-}(0) &= N\alpha^{-}
\end{aligned}
$$

where $\alpha^{-},\ \alpha^{+} \in \mathbb{R}_{+}^{4}$.

## Proposition

Let

$$X_1^+ = \frac{1}{K}(Z^+ - N_2X_2) \text{ and } X_1^- = \frac{1}{K}(Z^- - N_2X_2).$$

$X_1^-(t)$ and $X_1^+(t)$ define an observer interval of $X_1$ in the sens that for all $\alpha^- \leq X(0) \leq \alpha^+$.

$$X_1^-(t) \leq X_1(t) \leq X_1^+(t), \quad \forall t \geq 0.$$

The proof is based on the cooperative dynamical systems :

- Prove that $X_1^-(t) - X_1(t) < 0$ and $X_1^+(t) - X_1(t) > 0$ by invoking the monotony property of cooperative systems.

## Robust stabilization

Consider the pair of controls $D_1(t) = (1 + r)D(t)$ and $D_2(t) = \alpha W(t)$.

Goal : stabilize the output states : $Y = (y_1, y_2) = (S(t), D_o(t))$ around a suitable level $Y^d = (S^d, Do^d)$.

The output becomes :

$$
(\mathcal{S}r_3) : \begin{cases} \dfrac{dy_1}{dt} = -R_1(X_1, y) + D_1(t)(y_1^{in} - y_1) \\ \\ \dfrac{dy_2}{dt} = -R_2(X_1, y, D_1) + D_2(t)(y_2^{max} - y_2) \end{cases}
$$

with

$y_1^{in}(t) = \frac{S_{in}(t)}{1+r}$, $\quad y_2^{in}(t) = \frac{Do_{in}(t)}{1+r}$ $\quad$ and $\quad y_2^{max}(t) = Do_{max}$

but

$$R_1(X_1, y) = \frac{\mu(y)X_1}{Y_f} \; ; \quad R_2(X_1, y, D_1) = K_0 \frac{\mu(y)X_1}{Y_f} - D_1(y_2^{in} - y_2)$$

Hence, consider the upper and lower bound of $R_i$, $i = 1$

$$R_1^+(X_1^+, y) = \frac{\mu^+(t)X_1^+(t)}{Y_f}; \quad R_1^-(X_1^-, y) = \frac{\mu^-(t)X_1^-(t)}{Y_f}$$

and

$$R_2^+(X_1^+, y, D_1) = K_0 \frac{\mu^+(t)X_1^+(t)}{Y_f} - D_1(y_2^{in-} - y_2)$$

$$R_2^-(X_1^-, y, D_1) = K_0 \frac{\mu^-(t)X_1^-(t)}{Y_f} - D_1(y_2^{in+} - y_2)$$

## Theorem

The following feedback controls

$$\widetilde{D_1}(t, X_1^-, X_1^+, y) = \begin{cases} \dfrac{R_1^+(X_1^+, y) - \lambda_1^+ G_1(y_1)}{y_1^{in-} - y_1} & \text{if} \quad y_1 < y_1^d \\ \dfrac{R_1^-(X_1^-, y) - \lambda_1^- G_1(y_1)}{y_1^{in+} - y_1} & \text{if} \quad y_1 > y_1^d \end{cases} \tag{3.4}$$

and

$$\widetilde{D_2}(t, X_1^-, X_1^+, y) = \begin{cases} \dfrac{R_2^+(X_1^+, y, \widetilde{D_1}) - \lambda_2^+ G_2(y_2)}{y_2^{max-} - y_2} & \text{if} \quad y_2 < y_2^d \\ \dfrac{R_2^-(X_1^-, y, \widetilde{D_1}) - \lambda_2^- G_1(y_1)}{y_2^{max+} - y_2} & \text{if} \quad y_2 > y_2^d \end{cases} \tag{3.5}$$

stabilize exponentially the output $y$around $y^d$.

where

$$G_i(y_i) = \frac{1 - \exp^{-g_i(y_i)}}{y_i - y_i^d}; i = 1, 2$$

$$g_i(y_i) = \frac{1}{2}(y_i - y_i^d)^2$$

$\lambda_i^*$, ($i = 1, 2$ and $* = +, -$), are adjusting positive constants.

# Simulations

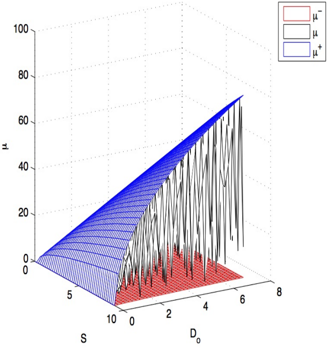



Figure – Kinetic $\mu$ with bounds.

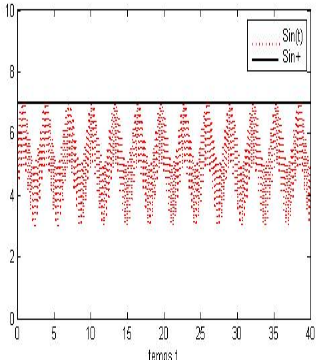



Figure – $S_{in}$ not well known and its bounds.

# Simulations

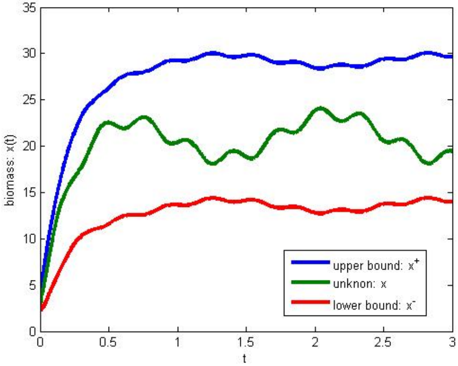



Figure – Upper and lower observer of $x$.

# Simulations

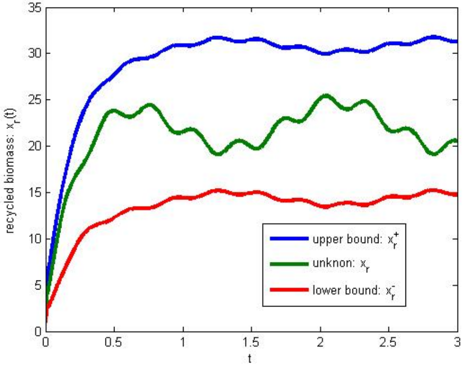



Figure – Upper and lower observer of $x_r$.

# Simulations

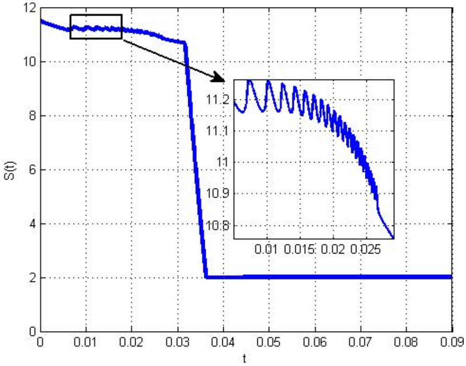



Figure – Stabilization towards $S^d = 2$.

# Simulations

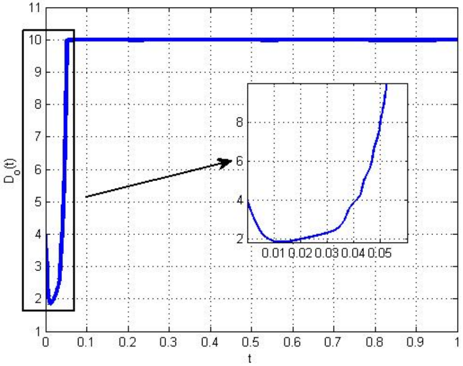



Figure – Stabilization towards $D_o^d = 10$.

## Model with time varying parameters

Let us come back to the anaerobic model $(S)$ and relax the assumptions by supposing that all parameters are a time varying functions

$$
(S_3) \quad \begin{cases}
\dot{s} = -\dfrac{\mu(t,s)x}{Y} - (1 + r(t))D(t)s + D(t)s_{in}(t) \\[2ex]
\dot{x} = \mu(t,s)x - (1 + r(t))D(t)x + r(t)D(t)x_r \\[2ex]
\dot{x}_r = \nu(1 + r(t))D(t)x - \nu(w(t) + r(t))D(t)x_r \\[2ex]
s(0) = s_0,\ x(0) = x_0,\ x_r(0) = x_{r0}, \\
s(T) = s_T,\ x(T) = x_T,\ x_r(T) = x_{rT}.
\end{cases}
$$

# New assumptions

$(\mathbf{H_1})$ :

- $\nu$ is a strictly positive constant.

- $D : \mathbb{R}^+ \to \mathbb{R}^+_*$ is Lebesgue measurable and there exist $\underline{D}, \overline{D} \in \mathbb{R}^+_*$ such that

$$\underline{D} < D(t) < \overline{D} \text{ for all } t \geq 0$$

- $r : \mathbb{R}^+ \to \mathbb{R}^+_*$ is Lebesgue measurable and there exist $\underline{r}, \overline{r} \in \mathbb{R}^+_*$ such that

$$\underline{r} < r(t) < \overline{r} \text{ for all } t \geq 0$$

- $w : \mathbb{R}^+ \to \mathbb{R}^+_*$ is Lebesgue measurable and there exists $0 < \overline{w} \in \mathbb{R}^+_*$ such that

$$w(t) < \overline{w} \text{ for all } t \geq 0$$

- ▷ $s_{in} : \mathbb{R}^+ \to \mathbb{R}^+_*$ is Lebesgue measurable and there exist $\overline{s}_{in} \in \mathbb{R}^+_*$ such that

$$0 < s_{in}(t) < \overline{s}_{in} \quad \text{for all} \quad t \geq 0$$

$(\mathbf{H_2})$ : $\mu : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is Lebesgue measurable w.r.t. the first variable and there exists $m > 0$ such that

$$\mu(t,s) \leq m \quad \text{for all } t \geq 0, \; s \geq 0.$$

$(\mathbf{H_3})$ : There exists $\phi : t \in \mathbb{R}^+ \to \phi(t) \in \mathbb{R}^+_*$ an integrable function such that

$$|\mu(t,s) - \mu(t,s')| \leq \phi(t)|s - s'| \; \forall \; s, s' \geq 0$$

# Controllability

Problem : The output of the activated sludge process station depends strongly on the influent rate of wastewater which changes according to several factors :

- the difference of flow between the night and day,
- between summer and winter,
- when it is raining or not.

Can one, using the control parameters, find a curve that stays "close" to a referential trajectory ?

To system $(S_3)$ we associate the following perturbed one :

$$
(S_3(v)) \quad \begin{cases}
\dot{s} = -\dfrac{\mu(t,s)x}{Y} - (1+r(t))D(t)s + D(t)s_{in}(t) \\[2ex]
\dot{x} = \mu(t,s)x - (1+r(t))D(t)x + r(t)D(t)x_r \\[2ex]
\dot{x}_r = \nu(1+r(t))D(t)x - \nu(w(t)+r(t))D(t)x_r \\[2ex]
s(0) = s_0 + v_{01},\ \ x(0) = x_0 + v_{02},\ \ x_r(0) = x_{r0} + v_{03} \\[2ex]
s(T) = s_T + v_{T1},\ x(T) = x_T + v_{T2},\ x_r(T) = x_{rT} + v_{T3}
\end{cases}
$$

where $v = (v_{01}, v_{02}, v_{03}, v_{T1}, v_{T2}, v_{T3}) \in \mathbb{R}^6$ and
$(r(t), D(t), w(t), s_{in}(t)) \in \mathcal{U}$

We fix a final time $T$. Let $W^1 := W([0, T], \mathbb{R}^3)$ be the space of absolutely continuous functions and $L^1 := L^1([0, T], \mathbb{R}^4)$.

We endow $L^1$ and $W^1$, respectively, with the norm

$$\|y\|_1 = \int_0^T \|y(t)\| dt \;\; ; \;\; \|y\|_{W^1} = \|y(0)\| + \int_0^T \|\dot{y}\| dt.$$

Consider the set of controls

$\mathcal{U} := \{(r, D, w, s_{in}) \in \mathbb{R}^4 : \underline{r} < r < \overline{r}, \underline{D} < D < \overline{D}, 0 < w < \overline{w}, 0 < s_{in} < \overline{s}_{in}\}$.

Let $\bar{y}(t) = (\bar{s}(t), \bar{x}(t), \bar{x}_r(t))$ the solution of $(S_3)$ associated to the control $\bar{u}(t) = (\bar{r}(t), \bar{D}(t), \bar{w}(t), \bar{s}_{in}(t))$ fulfilling

$$\bar{x}(t) + \bar{x}_r(t) > 0 \ \forall t \in [0, T]. \tag{4.1}$$

## Definition

The system $(S_3)$ is said to be strongly locally controllable at $(\bar{y}, \bar{u})$, if there exist $\alpha > 0$ and $\beta > 0$ such that for all $v \in \mathbb{R}^6$, with $\|v\| \le \beta$, there exist a trajectory $y = (s, x, x_r)$ and a control $u = (r, D, w, s_{in})$ for system $(S_3(v))$ such that

$$\|y - \bar{y}\|_{W^1} \le \alpha \|v\| \quad \text{and} \quad \|u - \bar{u}\|_1 \le \alpha \|v\|.$$

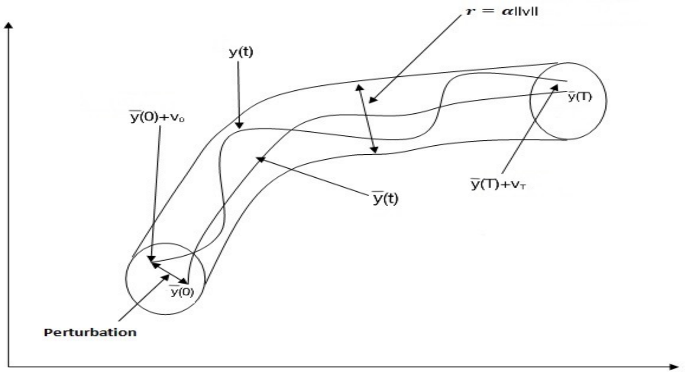



Figure – Strong controllability at $(\bar{y}, \bar{u})$.

# Local strong controllability

## Theorem

Let the assumptions $(H_1)$ and $(H_3)$ be satisfied and $(\bar{y}, \bar{u})$ be a solution of system $(S_3)$ satisfying $(H_1)$ and $(H_3)$. Then the system $(S_3)$ is strongly locally controllable at $(\bar{y}, \bar{u})$.

The proof is based on the nonsmooth analysis :

- For a dynamical system $(DS)$

$$
\begin{aligned}
\dot{Z}(t) &= f(Z(t), u(t)) \; \forall\, t \in [0, T] \\
(Z(0), Z(T)) &\in S := S_1 \times S_2, \\
u(t) &\in \mathcal{U}.
\end{aligned}
$$

Under some assumptions if $(DS)$ is Normal $\Longrightarrow$ $(DS)$ is Locally strongly controllable.

## Definition

The system $(DS)$ is called Normal at $Z$ if any arc $p \in W^1$ fulfilling

$$\begin{array}{rcl} \dot{p}(t) \in co\{q : (q,0) & \in & \partial^L[- < p(t), f(.,.) > + \Psi_{\mathcal{U}}(.)](Z(t), u(t))\} \; a.e. \\ < p(t), f(t, Z(t), u(t)) > & = & \max_{w \in \mathcal{U}} < p(t), f(t, Z(t), w) > \; a.e. \\ (p(0), p(T)) & \in & \partial^L d((Z(0), Z(T)); S), \end{array}$$

is trivial $(p = 0)$.

where $\Psi_{\mathcal{U}}(.)$ is the indicator function, $d((Z(0), Z(T)); S)$ is the metric from $(Z(0), Z(T))$ to $S$, and $\partial^L$ denotes the Limiting Fréchet subdifferential.

- prove that the system $(S_3)$ is Normal.

# Simulations

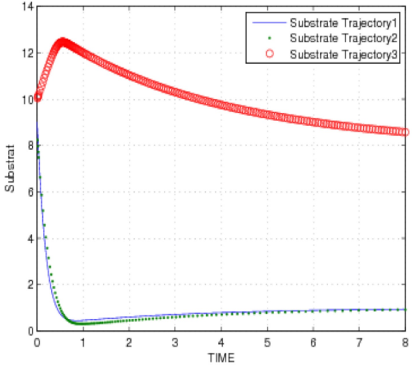



Figure – Local controllability of substrate.

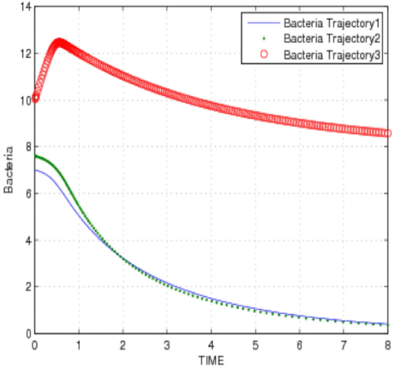


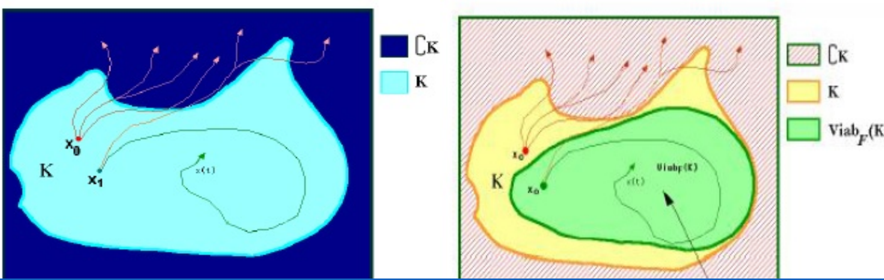

Figure – Local controllability of bacteria.

## Viability concept

We are now interested by the problem of the form

$$\begin{cases} \dfrac{dX}{dt} \in F(t, x(t)) \text{ for a.e. } t \geq 0 \\ X(t) \in K \ \ \forall t \geq 0; \ X(0) = X_0 \end{cases} \tag{4.2}$$

The goal is to analyze the existence of trajectories remaining in $K$.



# Characterization

## Definition

We say that a trajectory of (4.2) is viable in $K$ if for all $t \in [0, T]$, $x(t) \in K$,

the set $K$ is a viability domain for $F$ if for all $x_0 \in K$ there exists a trajectory $x(.)$ of system (4.2) such that $x(t) \in K \; \forall \; t \in [0, T]$ with $x(0) = x_0$.

# Viability



## Theorem

Suppose that $F$ is a Marchaud set valued map and $K$ a closed subset, then the following equivalences hold :

- $K$ is a viability domain for $F$.
- $F(x) \cap T_K(x) \neq \emptyset, \ \ \forall\, x \in K$.

where $T_K(x) =$ is the contingent cone (Bouligand cone) of $K$.

$$T_K(x) = \{w \in x : \liminf_{t \searrow 0} \frac{dist(x + tw, K)}{t} = 0\}$$

## Viability in the wastewater treatment

The activated sludge model $(S)$ can be rewritten as a general nonlinear dynamical system

$$\begin{cases} \dfrac{dX}{dt} = f(t, X(t), u(t)) := [A(u(t)) + \mu(t, X)B]X + W_u(t) \text{ for a.e. } t \geq 0 \\ X(0) = X_0 = (s_0, x_0, x_{r0})^T; \; u = ((1+r)D, rD, (w+r)D) \in \mathcal{U}. \end{cases} \quad (4$$

So, by choosing

$$F(t, X(t)) := \{f(t, X(t), u(t)) \; : \; u(t) \in \mathcal{U}\}$$

and considering the viability set

$$K(t) := a(t) + D \;\; \forall \; t \geq 0,$$

the activated sludge model $(S)$ becomes the viability problem (4.2).

$$\begin{cases} \dfrac{dX}{dt} \in F(t, x(t)) \text{ for a.e. } t \geq 0 \\ X(t) \in K(t) \quad \forall t \geq 0; \\ X(0) = X_0 \end{cases} \tag{4.4}$$

where the functions governing this system fulfill

$u \,:\, \mathbb{R}^+ \longrightarrow \mathcal{U} \subset \mathbb{R}^n_+$ measurable with value in a compact set $\mathcal{U}$.

$A \,:\, \mathbb{R}^n \longrightarrow \mathcal{M}(\mathbb{R}^n)$ is a linear operator.

$\mu \ : \ \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^+$ bounded, $t-$measurable and $x-$Lipschitz.

$W_u \ : \ R_+ \longrightarrow \mathbb{R}^n_+$ measurable, bounded and Lipschitz with respect $u$.

$B$ is a real constant matrix and $X_0 \in \mathbb{R}^n$.

$a : \mathbb{R}^+ \longrightarrow a(t) \in \mathbb{R}^n$ is an absolutely continuous mapping.

$D$ is a closed subset of $\mathbb{R}^n$.

# Necessary and sufficient condition of viability

## Theorem

Let $T > 0$. The following assertions are equivalent

1. There exists a set $I \subset [0, T]$ of full measure such that

$$\forall t \in \ I, \forall X \in \ K(t), \ f(t, X) - \dot{a}(t) \in \overline{co} T_B(D, x - a(t))$$

2. $\forall \ t_0 \in [0, T]$ and $X_0 \in K(t_0)$, the problem (4.3) admit a viable solution $X(\cdot)$.

# Viability

Viability

The proof is based on

- Prove that $F$ is a Marchaud set valued map
- Prove that $K(\cdot)$ is an absolutely continuous set valued map
- Prove the fact that

$$f(t,X) - \dot{a}(t) \in \overline{co}T_B(D, x - a(t)) \Leftrightarrow (1, f) \in coT_B(graphe(K), (t, x))$$

and use the viability theorem.

## Particular case

For the problem $(S)$ , by choosing

$$D \equiv \mathbb{R}_+^3$$

and

$$a(t) = (\alpha t, \beta t, \gamma t)$$

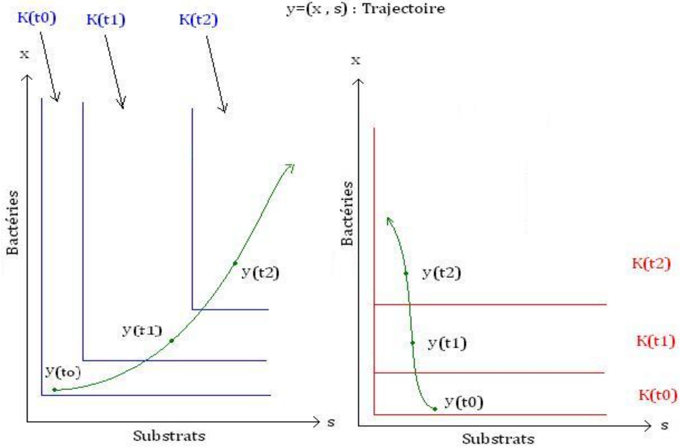
with $\alpha$, $\beta$ and $\gamma$ positives constants

or

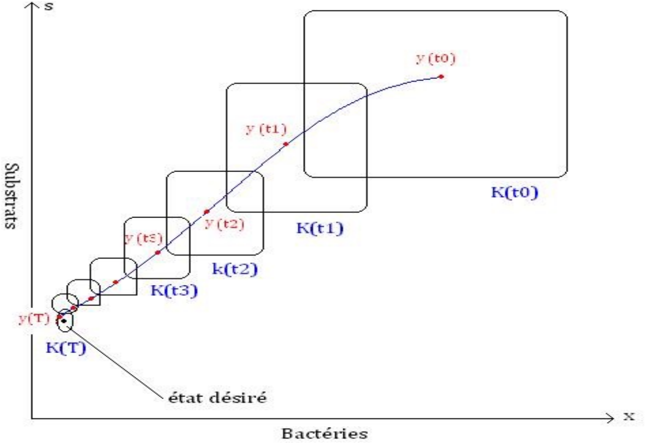
$$a(t) = (\alpha, \beta t, \gamma t)$$

we obtain the following schematics :

## Simulations



## Under working : control by viability



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## Capture Basin

Consider a closed subset $C$ as a target of system $(S)$.

### Definition

The capture basin of $C$ is the set of initial states $x$ from which starts at least one solution of $(S)$ reaching the target at a time $T$.
$Capt(C) = \{x \in \mathbb{R}^n : \exists\, x(.)$ sol'n of $(S)$ with $x(0) = x$, s.t. $x(T) \in C\}$

## Viable-Capture Basin

If the target $C \subset K$.

### Definition

The Viable-capture basin of $C$ in $K$ is the set $Capt^K(C)$ of initial states $x \in K$ from which starts at least one evolution $x(.)$ of $(S)$ viable in $K$ on $[0, T[$ until a finite time $T$ when the evolution reaches the target at $x(T) \in C$.

$$Capt^K(C) = \{x \in K : \exists\, x(.) \text{ solution of } (S) \text{ with } x(0) = x,$$

$$x(t) \in K,\ \forall\, t \in [0, T[ \text{ and } x(T) \in C\}$$

# Viability kernel of $K$ with target $C$

### Definition

The Viability kernel of $K$ with target $C$ of initial states $x \in K$ such that at least one solution $x(.)$ of $(S)$ starting at $x_0$ is viable in $K$ for all $t \geq 0$ or viable in $K$ until it reaches $C$ in finite time, $Viab(K, C)$.

$$Viab(K, C) = \{x \in K : \ \exists\, x(.) \text{ solution of } (S) \text{ with } x(0) = x,$$

$$x(t) \in K, \ \forall\, t \geq 0 \text{ or } x(t) \in K, \ \forall\, t \in [0, T[ \text{ and } x(T) \in C\}$$

# Bibliographies

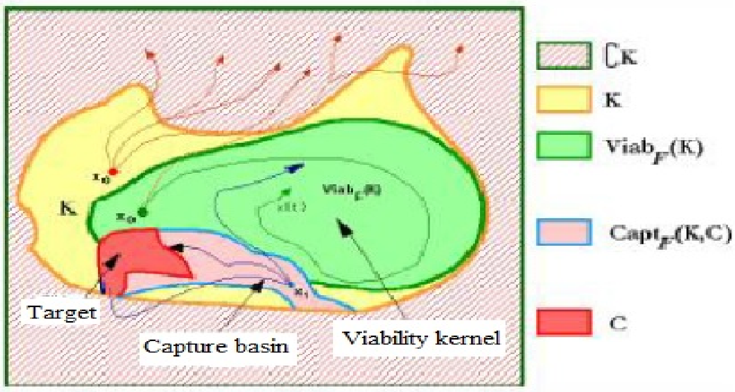


Figure – Capture basin

# Characterization

## Property

$$Viab(K, C) = Viab(K \setminus C) \cup Capt^{K}(C)$$

## Theorem

Assume that $F$ is Marchaud and that the target $C \subset K$ and $K$ are closed. The viability kernel $Viab(K, C)$ of the subset $K$ with target $C$ satisfies

- $Viab(K, C)$ is the largest closed subset $D$ satisfying $C \subset D \subset K$.
- $D \setminus C$ is locally viable under $F$ :

$$\forall\, x \in D \setminus C,\ \ F(x) \cap T_D(x) \neq \emptyset$$

## Link with HJB equation (H. Zidani et al.)

Let

$$S_{[0,t]}(x) = \{x(.) \text{ sol'n of } (S) \text{ s.t. } x(0) = x\}$$

and consider a continuous functions $g : \mathbb{R}^n \to \mathbb{R}$ and $v_0 : \mathbb{R}^n \to \mathbb{R}$, such that

$$g(x) \leq 0 \Leftrightarrow x \in K \text{ and } v_0(x) \leq 0 \Leftrightarrow x \in C$$

Then we introduce the new control problem

$$V(t,x) = \inf_{x(.)\in S_{[0,t]}(x)} \{\max(v_0(x(t))), \max_{\theta\in[0,t]} (g(x(\theta)))\}$$

## Link with HJB equation (H. Zidani et al.)

Define the Hamiltonian $H(x,p) = \max_{q \in F(x)}(-p.q)$. We have :

**Theorem**

$V(t,x)$ is the unique continuous viscosity solution of the HJB equation

$$\min\{\partial V_t(t,x) + H(x, D_x V(t,x)), V(t,x) - g(x)\} = 0,$$

with $V(0,x) = \max(v_0(x), g(x))$.

$\forall\ (t,x) \in [0,+\infty[ \times \mathbb{R}^n$.

## Link with HJB equation (H. Zidani et al.)

Using the value function $V$, we can characterize the capture basin with target as :

### Lemma

We have

$$Cap^K(C,t) = \{x \in \mathbb{R}^n, V(t,x) \leq 0\}.$$