Retour sur l’épidémiologie

$S.I.R.$

$$\left\{\begin{array}{rcl} \dfrac{dS}{dt} & = & -\beta SI \\ \dfrac{dI}{dt} & = & +\beta SI-(\gamma+\mu)I \\ \dfrac{dR}{dt} & = & \gamma I \end{array}\right.$$

Plusieurs sites

Coefficients dépendants du temps

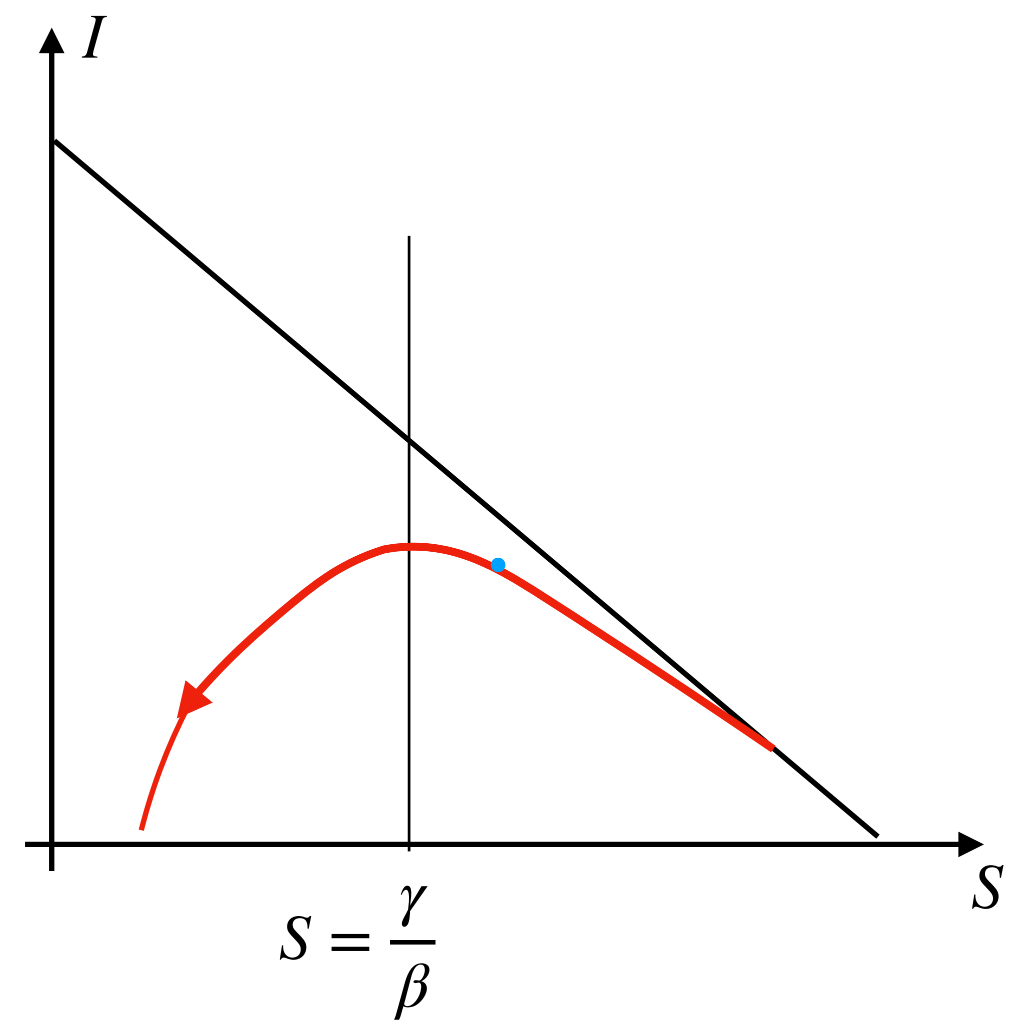
Deux sites + coefficients périodiques

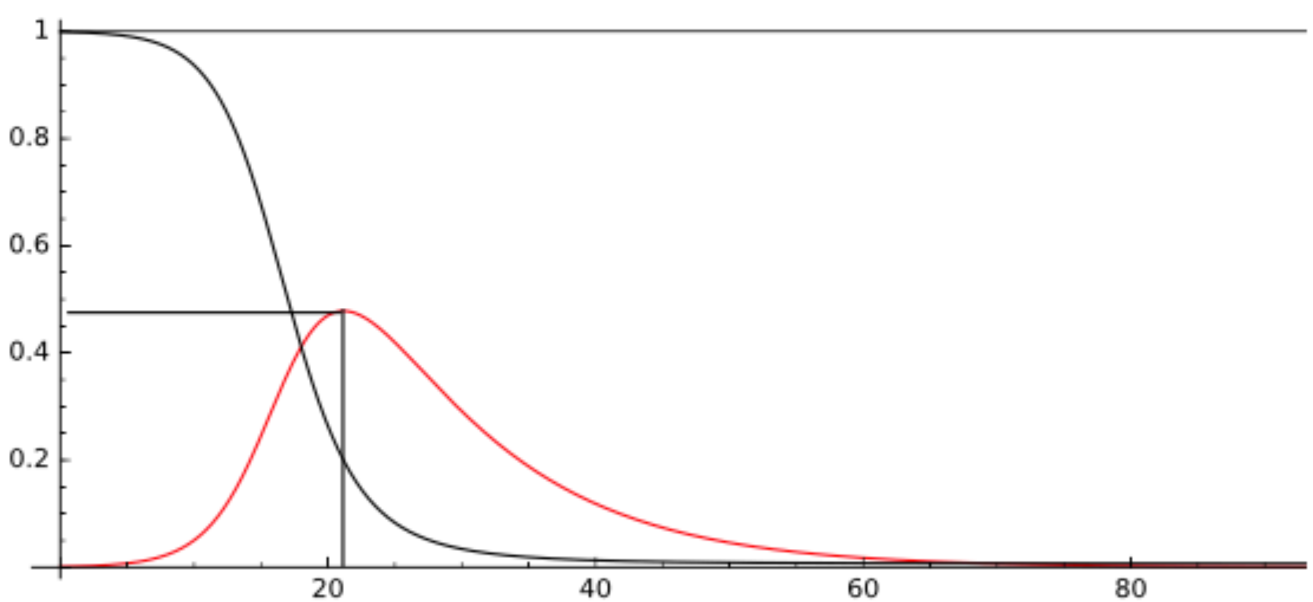
Rappel sur SIR (pour fixer les idées)

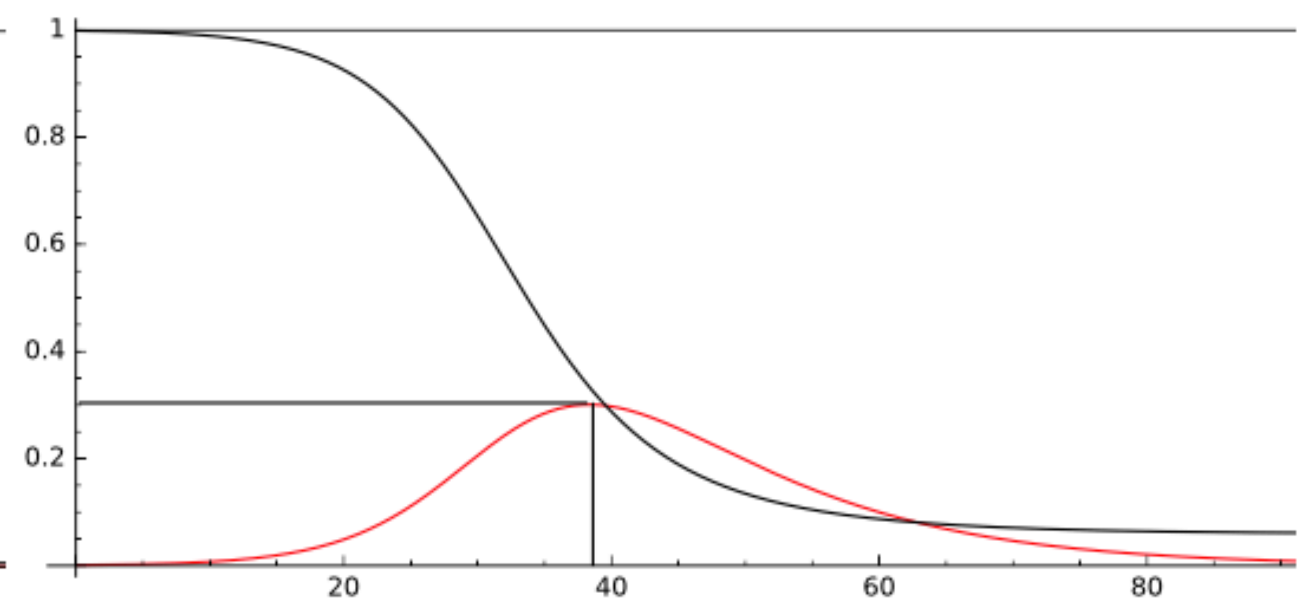
S.I.R. $$\begin{cases} \frac{dS}{dt} &= -\beta SI \\ \frac{dI}{dt} &= +\beta SI - (\gamma+\mu)I \\ \frac{dR}{dt} &= \gamma I \end{cases} \quad = I\big(\beta S - (\gamma + \mu)\big)$$

$\mu \approx 0$

$$S + I - \frac{\gamma}{\beta}\log(S) = \text{constante}$$





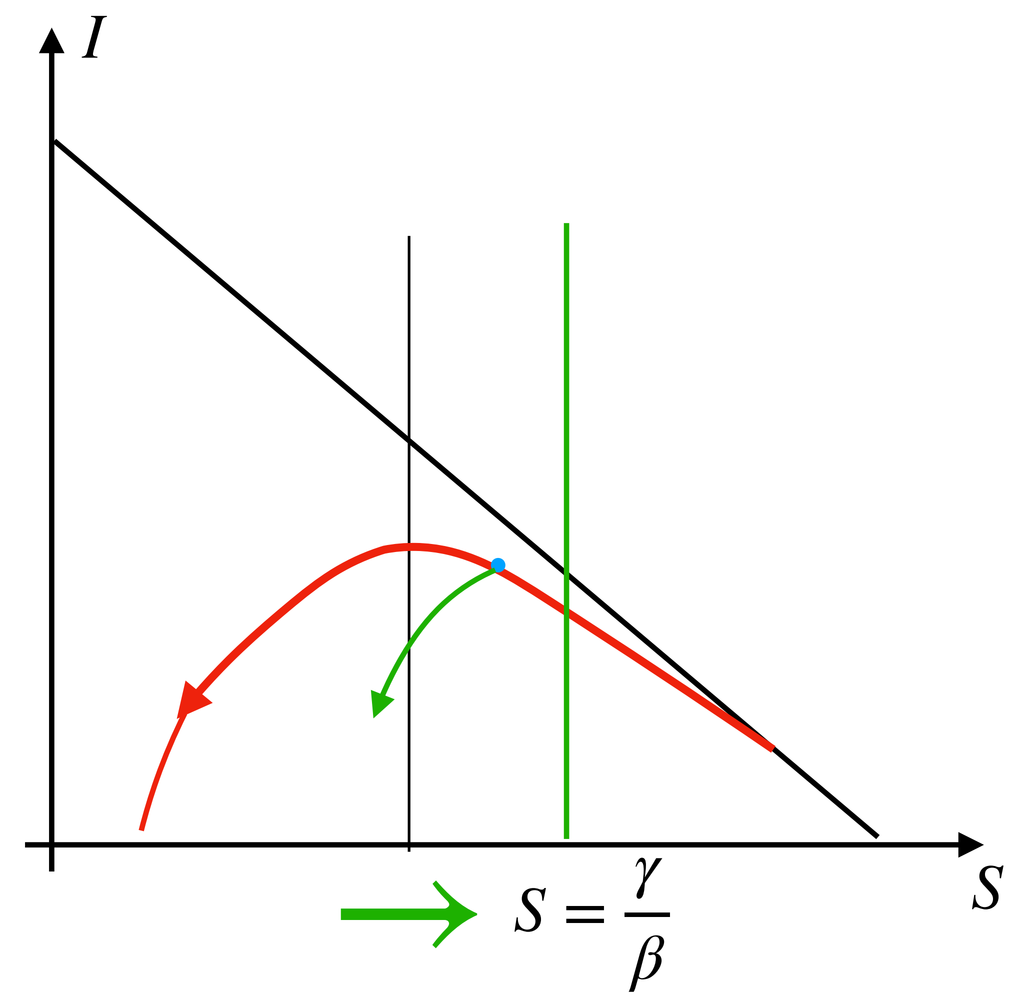


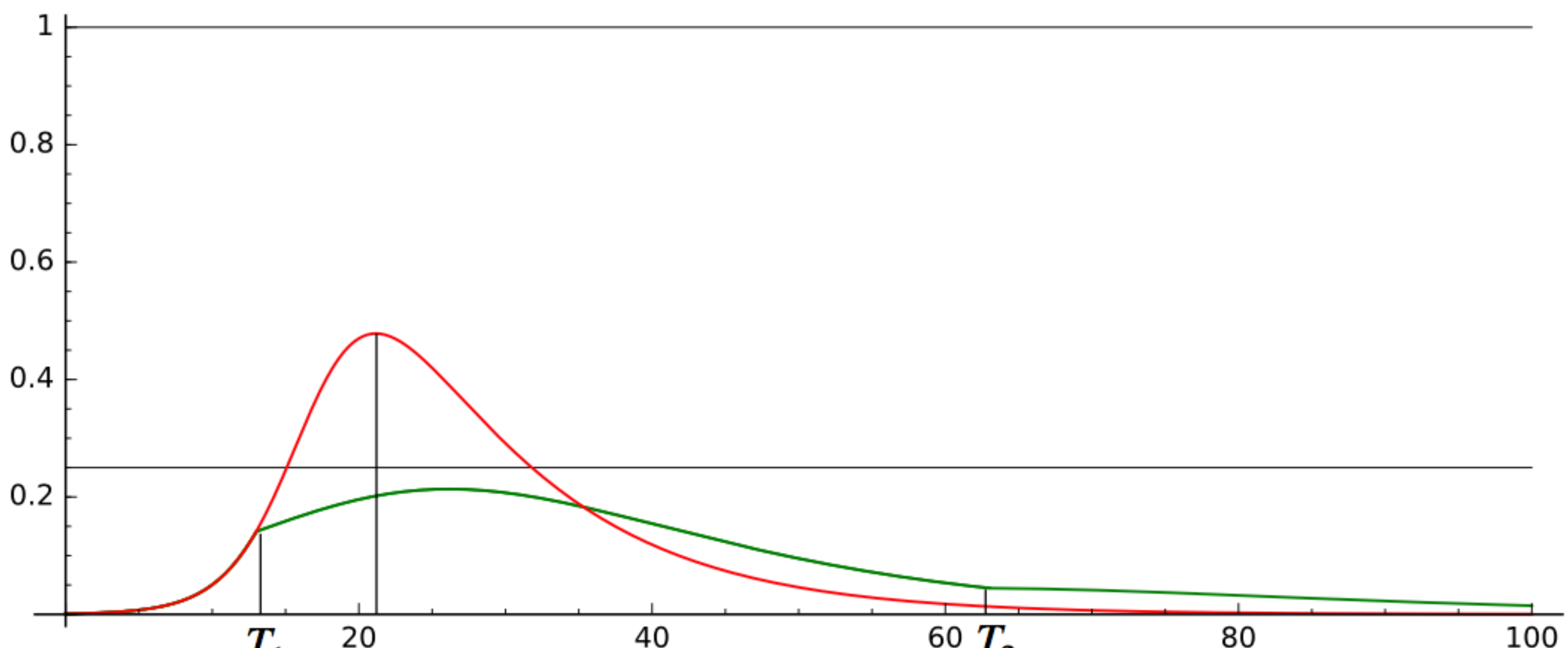
$$
\text{S.I.R.} \qquad \begin{cases} \dfrac{dS}{dt} = -\beta S I \\[4pt] \dfrac{dI}{dt} = +\beta S I - (\gamma + \mu) I \\[4pt] \dfrac{dR}{dt} = \gamma I \end{cases} \qquad = I\big(\beta S - (\gamma + \mu)\big)
$$

$\mu \approx 0$

$$
S + I - \frac{\gamma}{\beta}\log(S) = \text{constante}
$$

Confinement : diminue β et augmente γ





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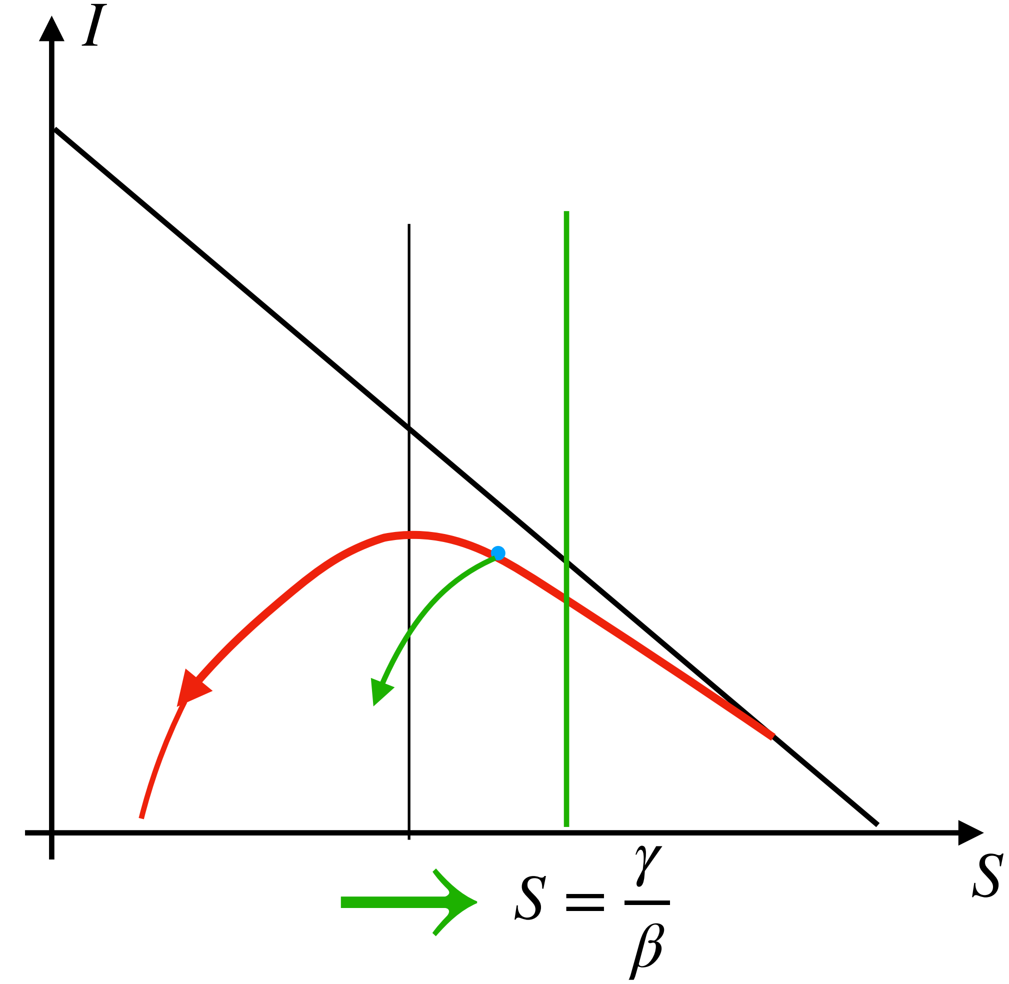
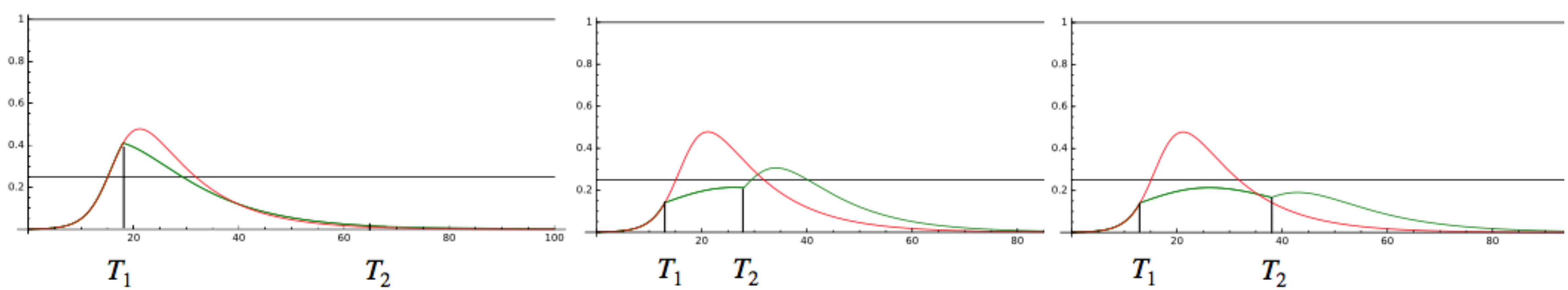
$\mu \approx 0$

$S + I - \dfrac{\gamma}{\beta}\log(S) = \text{constante}$

Confinement : diminue β et augmente γ





Deux sites $i = 1,2$

$$\frac{dS_i}{dt} = -S_i I_i \beta_i(t)$$

$$\frac{dI_i}{dt} = S_i I_i \beta_i(t) - \left[\gamma_i(t) + \mu\right] I_i$$

$$\frac{dR_i}{dt} = \gamma_i(t) I_i$$

$$i = 1,2 \qquad j = i + 1 \pmod 2$$

Le modèle SIR standard

Deux sites $i = 1,2$

$$\frac{dS_i}{dt} = -S_i I_i \beta_i(t) - mS_i + mS_j$$

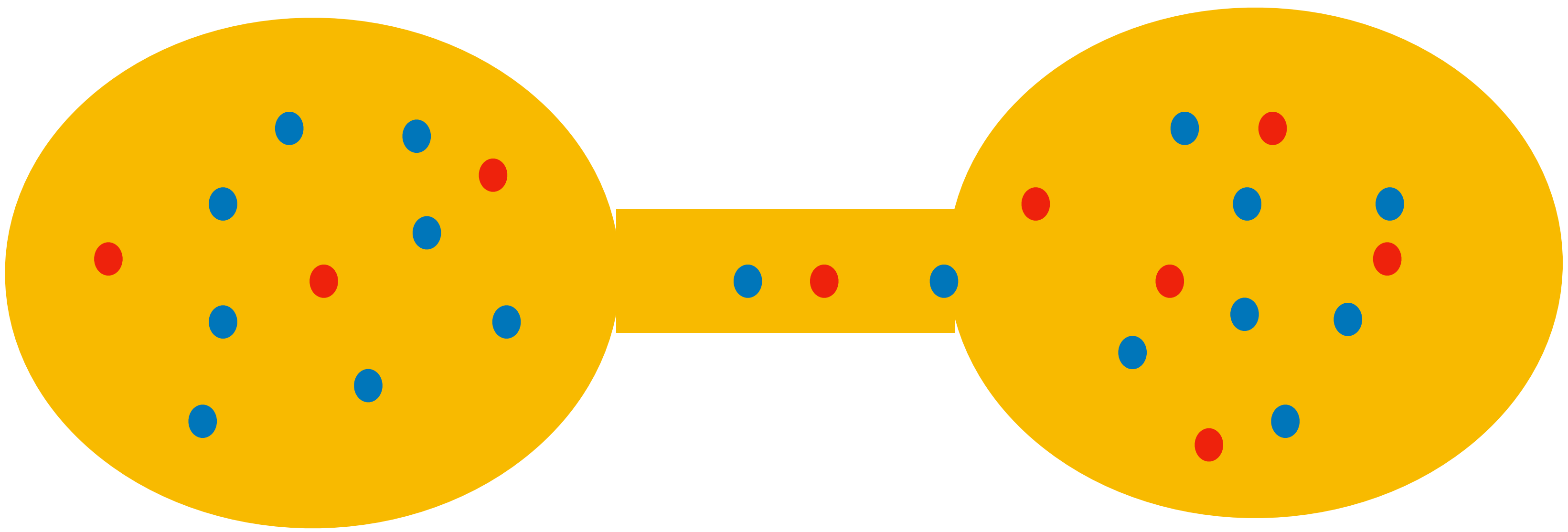
$$\frac{dI_i}{dt} = S_i I_i \beta_i(t) - \left[\gamma_i(t) + \mu\right] I_i - mI_i + mI_j$$

$$\frac{dR_i}{dt} = \gamma_i(t) I_i - mR_i + mR_j$$

Le modèle SIR standard

+ Migration

$i = 1,2 \qquad j = i + 1 (\mathrm{mod}\, 2)$



Deux sites $i = 1,2$

$$\frac{dS_i}{dt} = -S_i I_i \beta_i(t) - mS_i + mS_j$$

$$\frac{dI_i}{dt} = S_i I_i \beta_i(t) - \left[\gamma_i(t) + \mu\right] I_i - mI_i + mI_j \, .$$

$$\frac{dR_i}{dt} = \gamma_i(t) I_i - mR_i + mR_j$$

Le modèle SIR standard

+ Migration

Même politique sanitaire sur chaque site

$$\beta_1(t) = \beta_2(t) \qquad \gamma_1(t) = \gamma_2(t)$$

Deux sites $i = 1{,}2$

$$\frac{dS_i}{dt} = -S_i I_i \beta_i(t) - mS_i + mS_j$$

$$\frac{dI_i}{dt} = S_i I_i \beta_i(t) - \left[\gamma_i(t) + \mu\right] I_i - mI_i + mI_j \,.$$

$$\frac{dR_i}{dt} = \gamma_i(t) I_i - mR_i + mR_j$$

Le modèle SIR standard

+ Migration

Même politique sanitaire sur chaque site

$$\beta_1(t) = \beta_2(t) \qquad \gamma_1(t) = \gamma_2(t)$$

Mais déphasée

$$\beta_1(t+\varphi) = \beta_2(t) \qquad \gamma_1(t+\varphi) = \gamma_2(t)$$

$$\frac{dS_i}{dt} = -S_i I_i \beta_i(t) - mS_i + mS_j$$
$$\frac{dI_i}{dt} = S_i I_i \beta_i(t) - \left[\gamma_i(t) + \mu\right] I_i - mI_i + mI_j .$$
$$\frac{dR_i}{dt} = \gamma_i(t) I_i - mR_i + mR_j$$

Paramètres hors mesures sanitaires

$$\beta_i^- = 0.1988$$
$$\gamma_i^- = 0.098$$
$$\mu_i^- = 0.002$$

Paramètres avec mesures sanitaires

$$\beta_i^+ = 0.0288$$
$$\gamma_i^+ = 0.128$$
$$\mu_i^+ = 0.002$$

	$\left[0\ ,\ \frac{T}{2}\right]$	$\left[\frac{T}{2}\ ,\ T\right]$
Site n°1	$+$	$-$
Site n°2	$-$	$+$

$$I_i(0) = 0.001$$
$$S_i(0) = 0.999$$

$1 = 10^6$ **individus**

Ce sont les paramètres de Holt et al.

$$\frac{dS_i}{dt} = -S_i I_i \beta_i(t) - mS_i + mS_j$$

$$\frac{dI_i}{dt} = S_i I_i \beta_i(t) - \left[\gamma_i(t) + \mu\right] I_i - mI_i + mI_j .$$

$$\frac{dR_i}{dt} = \gamma_i(t) I_i - mR_i + mR_j$$

	$\left[0 \quad , \quad \frac{T}{2}\right]$	$\left[\frac{T}{2} \quad , \quad T\right]$
Site n°1	$+$	$-$
Site n°2	$-$	$+$

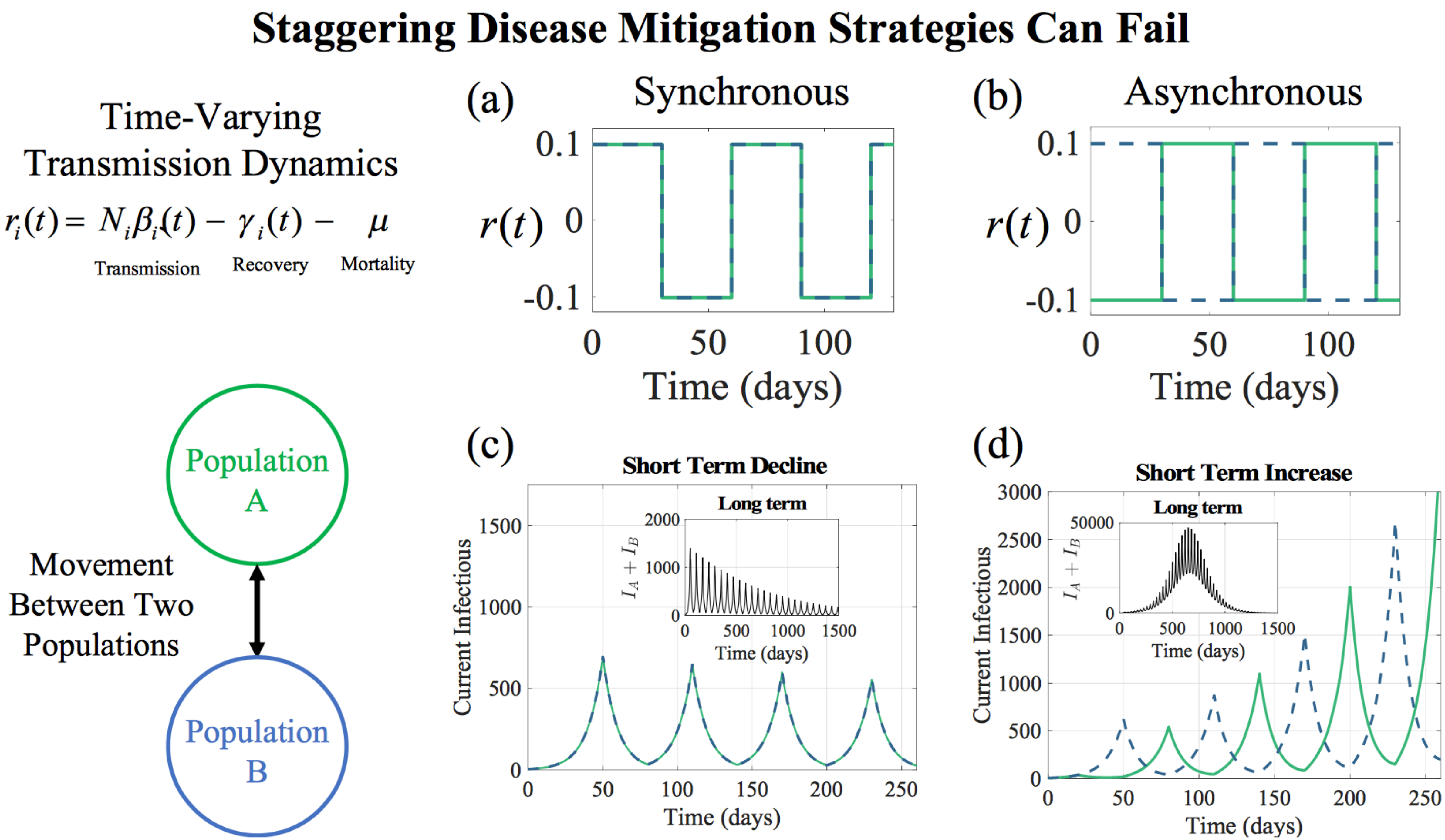
Site 1 = vertueux : confine tout de suite

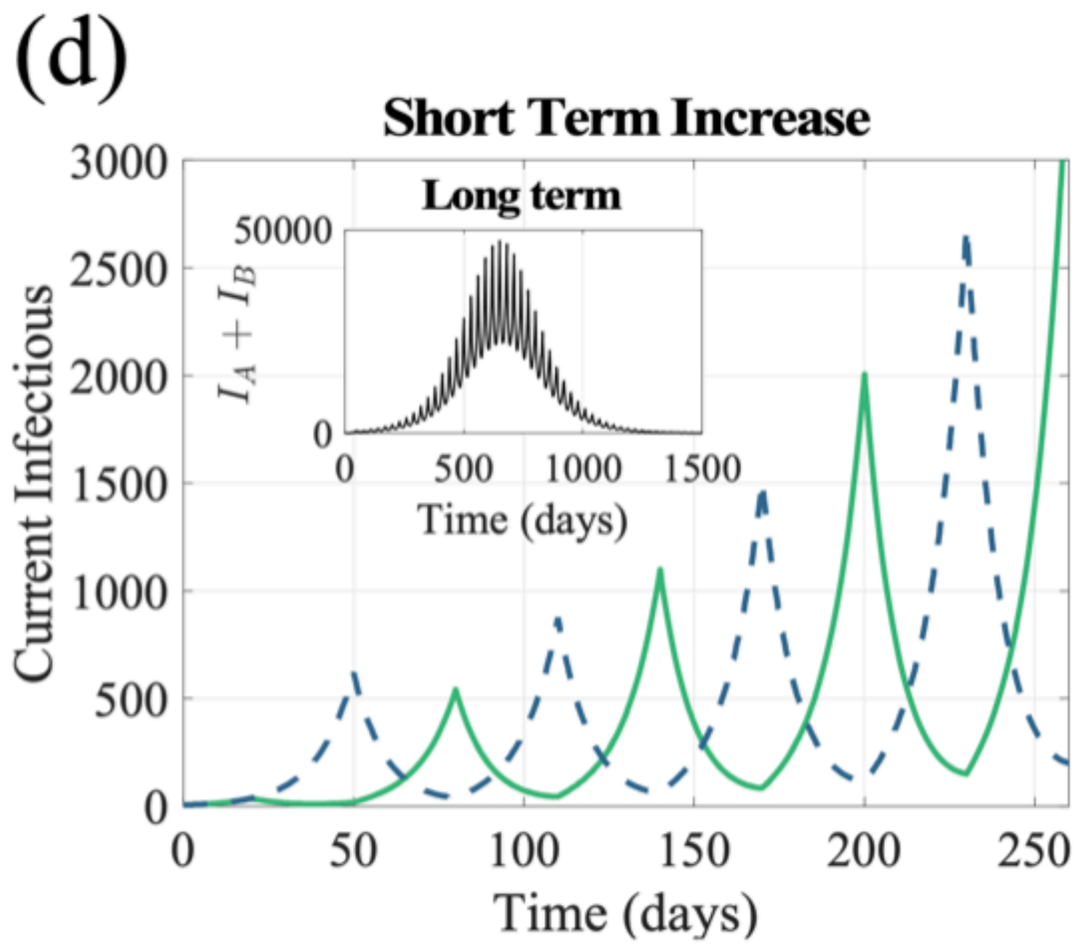
Site 2 = laxiste : tarde à confiner

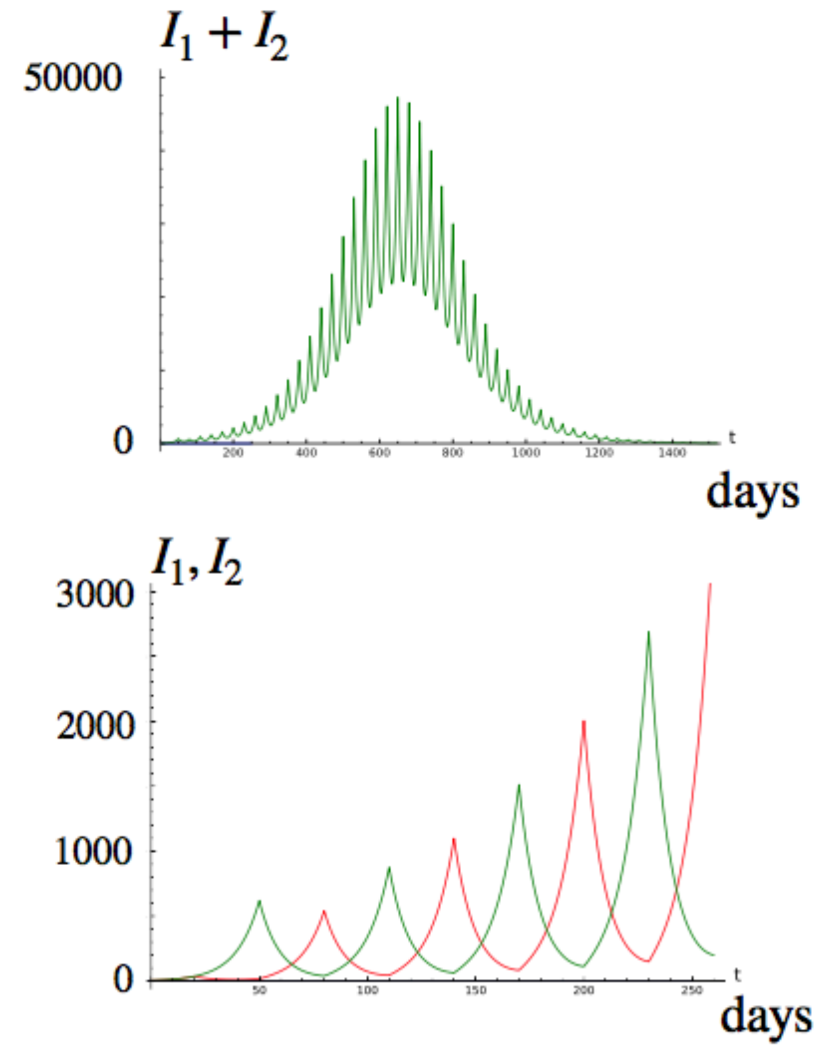
Paramètres à explorer : T et m

Les articles KSBGSH

Figures







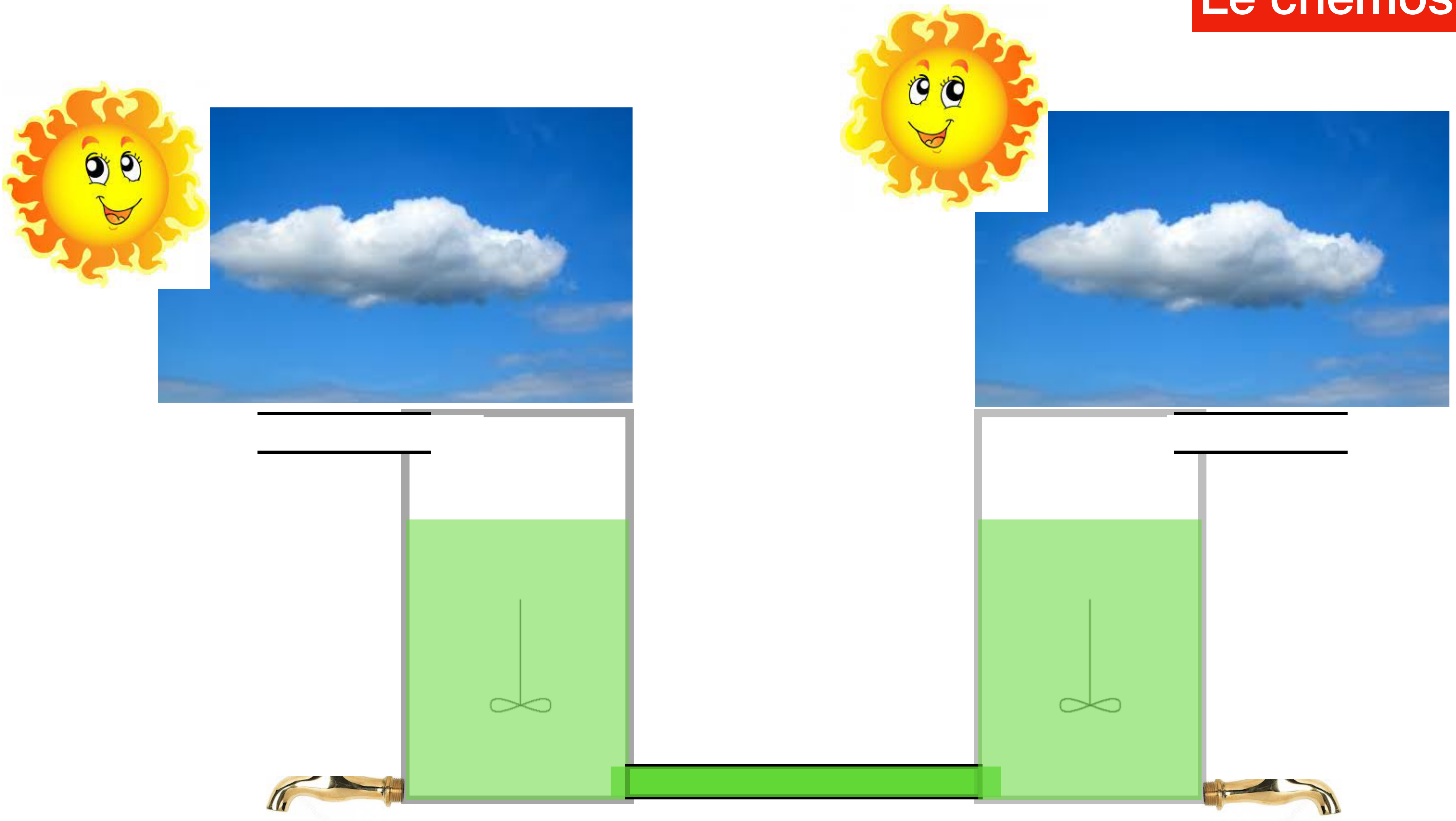
Même modèle simulé par moi

$\beta_n N = 0.1988$	$\gamma_n = 0.098$	$\mu_n = 0.002$
$\beta_s N = 0.0288$	$\gamma_s = 0.128$	$\mu_s = 0.002$

$\beta_n N - (\gamma_n + \mu_n) = 0.0988$
$\beta_s N - (\gamma_s + \mu_s) = -0.1012$

Parenthèse sur le chémostat

Le chémostat

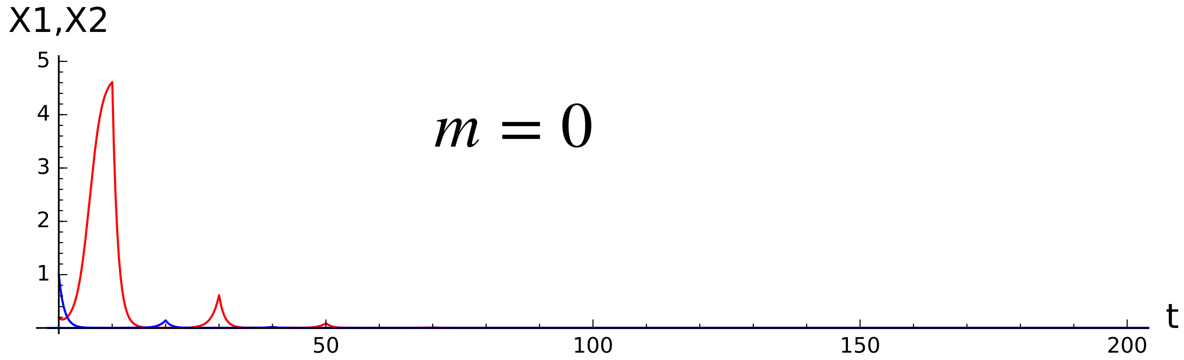


Le chémostat

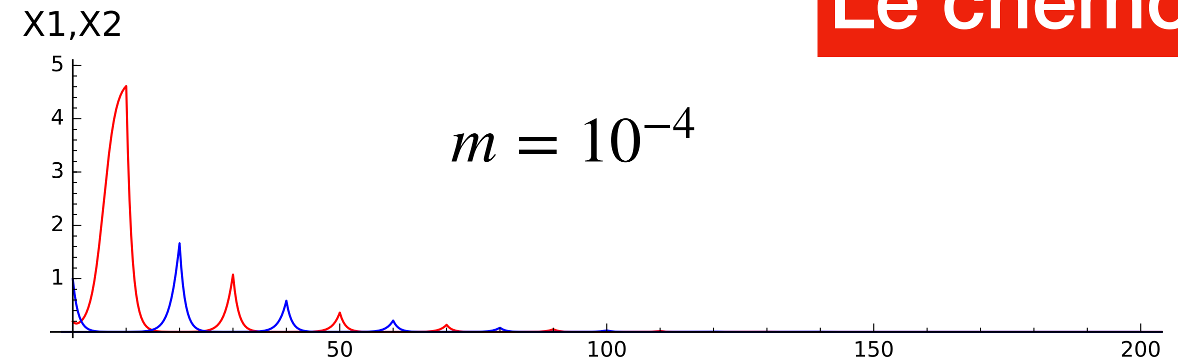
$$
\left\{
\begin{array}{lcl}
\dfrac{ds_1}{dt} & = & D(S_{in} - s_1) - r_1(t)\mu(s)x_1 + m(s_2 - s_1) \\
\dfrac{dx_1}{dt} & = & (r_1(t)\mu(s_{)} - D)x_1 + m(x_2 - x_1) \\
\dfrac{ds_2}{dt} & = & D(S_{in} - s_2) - r_2(t)\mu(s)x_2 + m(s_1 - s_2) \\
\dfrac{dx_2}{dt} & = & (r_2(t)\mu(s_2) - D)x_2 + m(x_1 - x_2)
\end{array}
\right.
$$

Le chémostat

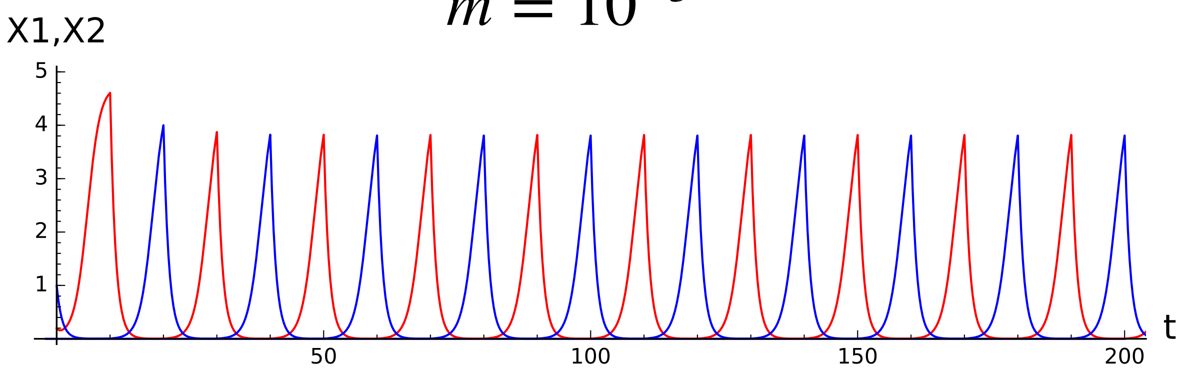
$m = 0$



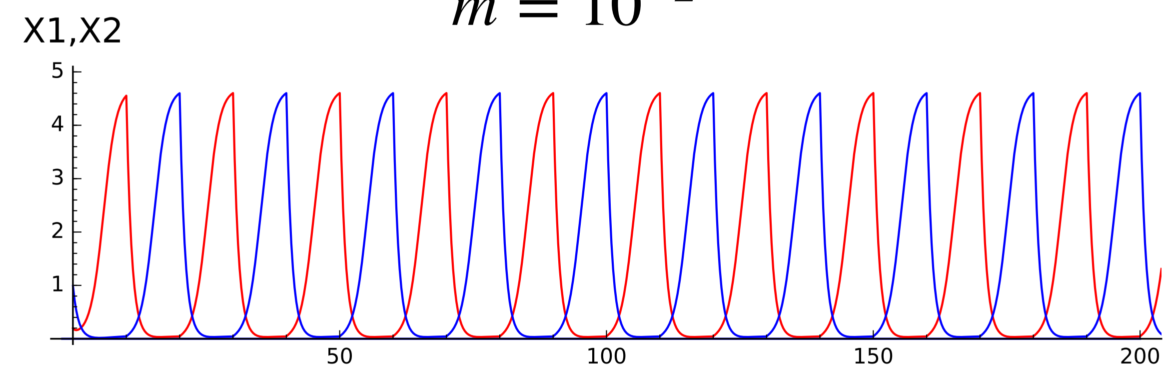
$m = 10^{-4}$



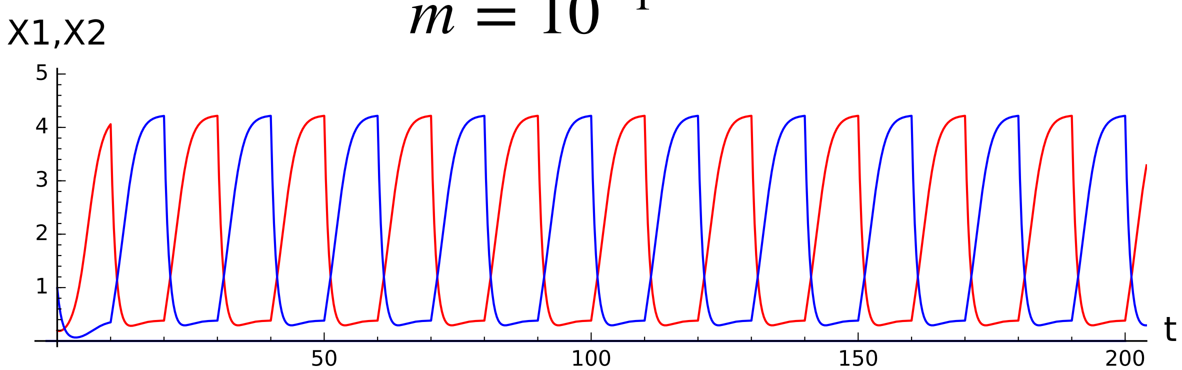
$m = 10^{-3}$



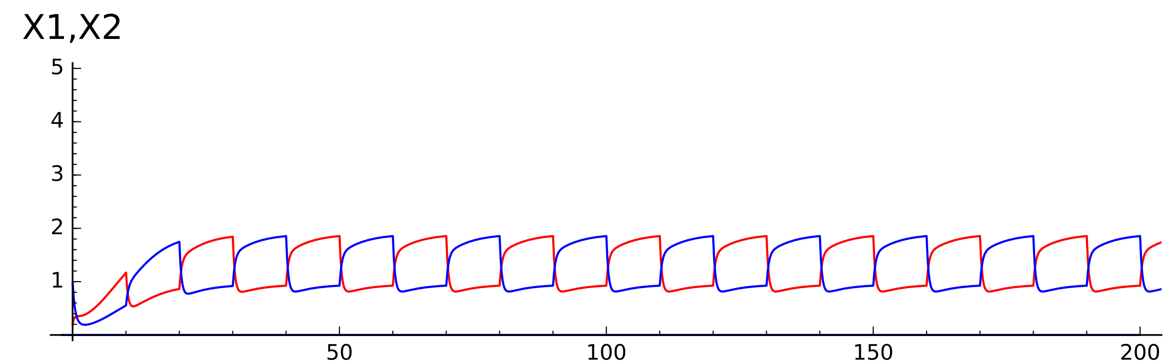
$m = 10^{-2}$



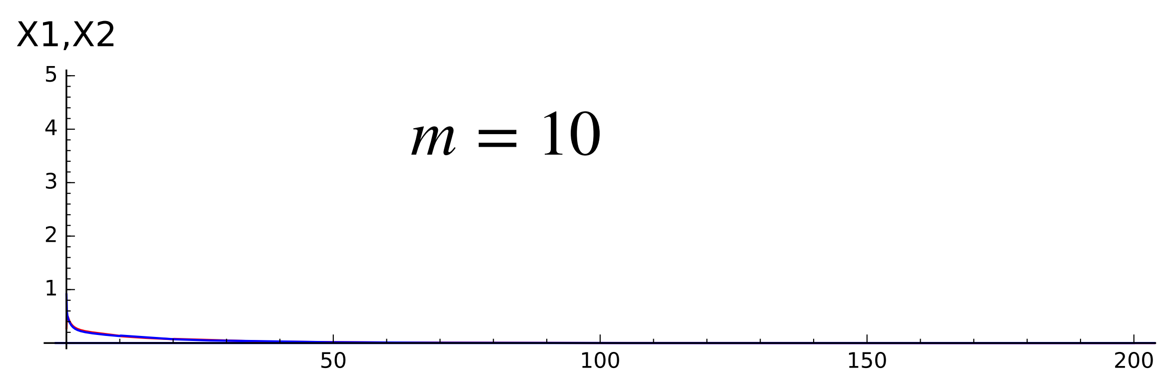
$m = 10^{-1}$



$m = 1$



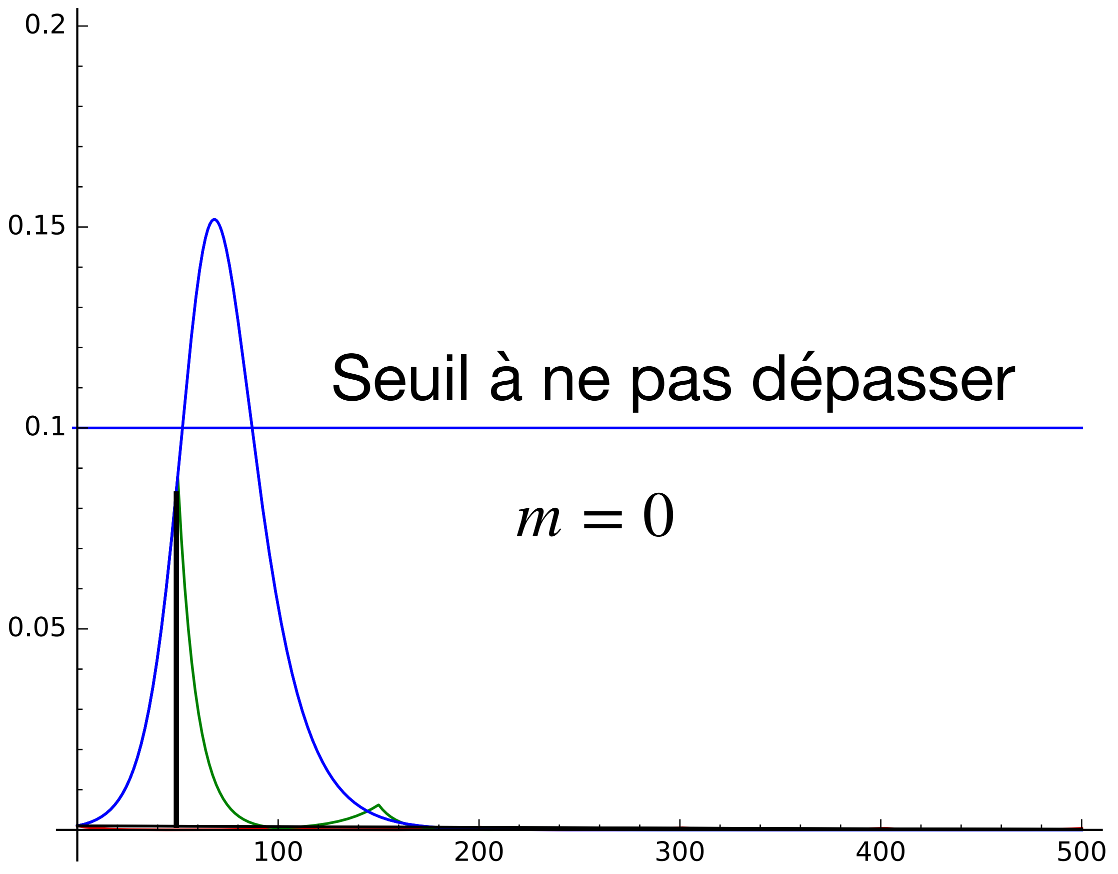
$m = 10$



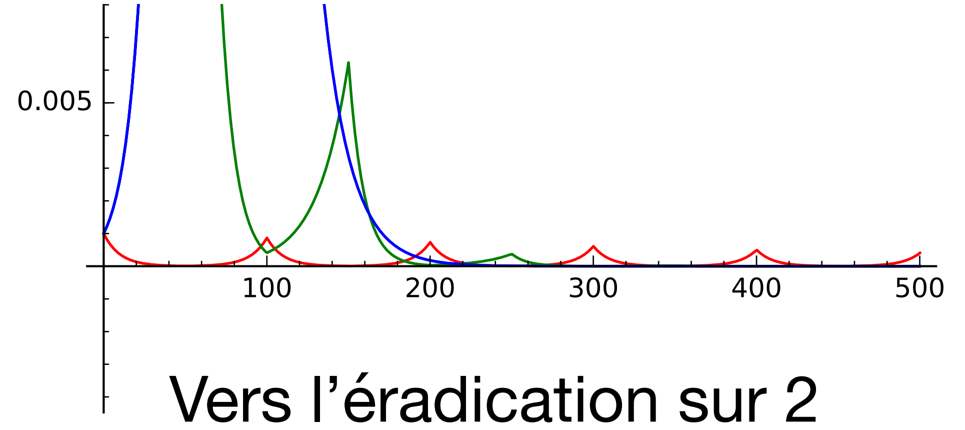
Fin parenthèse

T = 100

Modèle diffusion



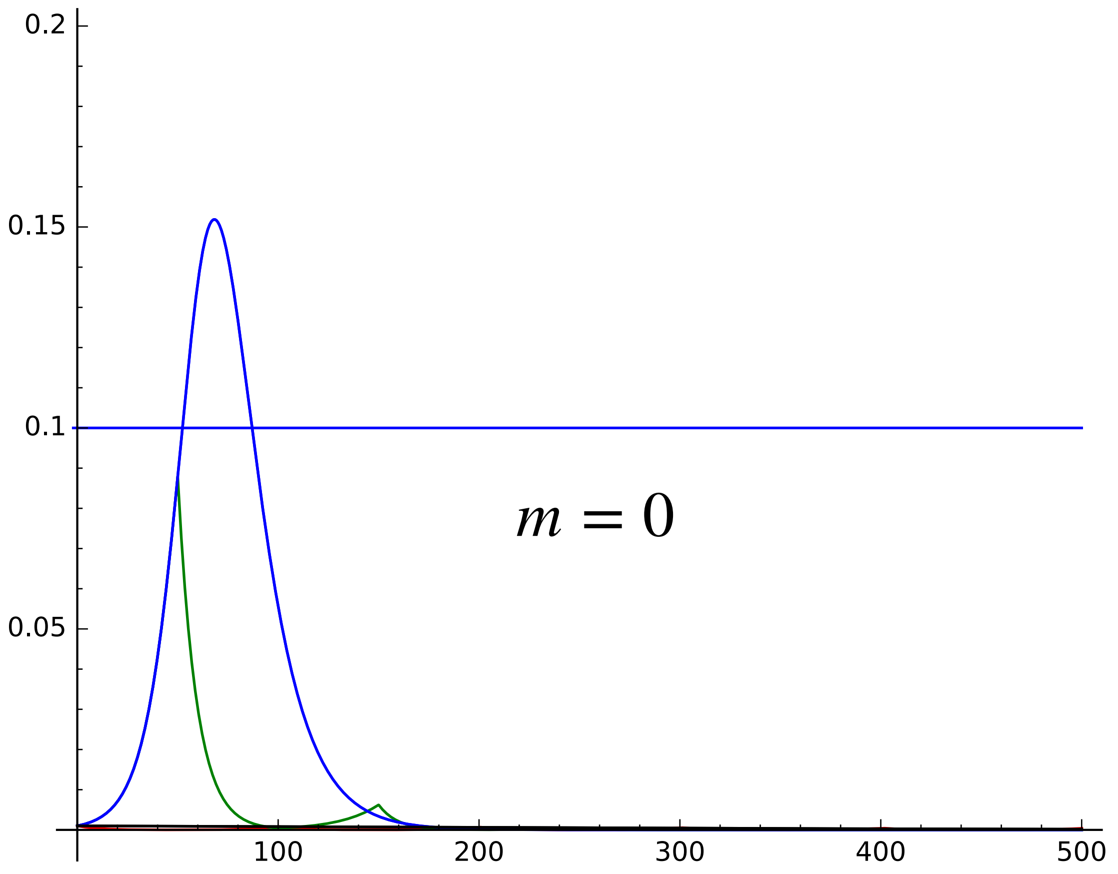
Politique tardive sur 1 : passe de justesse

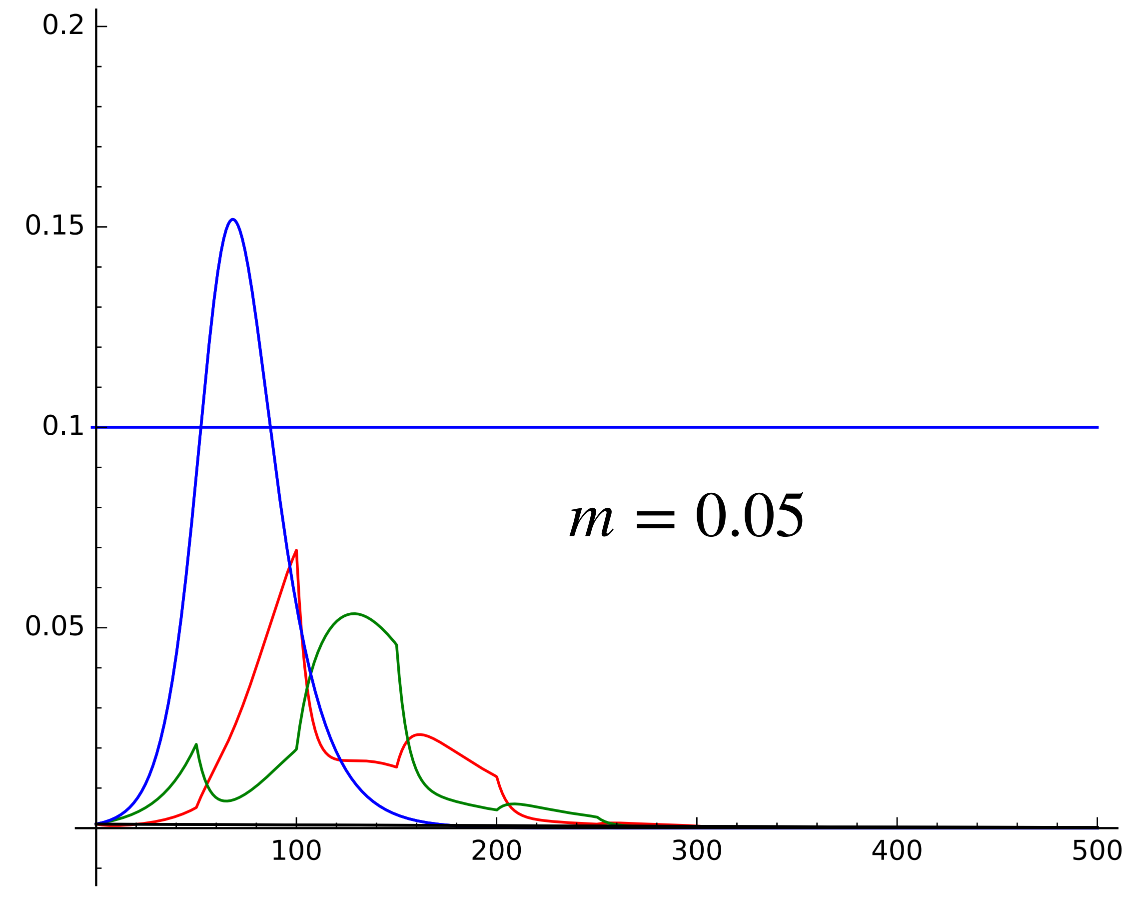


Vers l'éradication sur 2

T = 100

Modèle diffusion

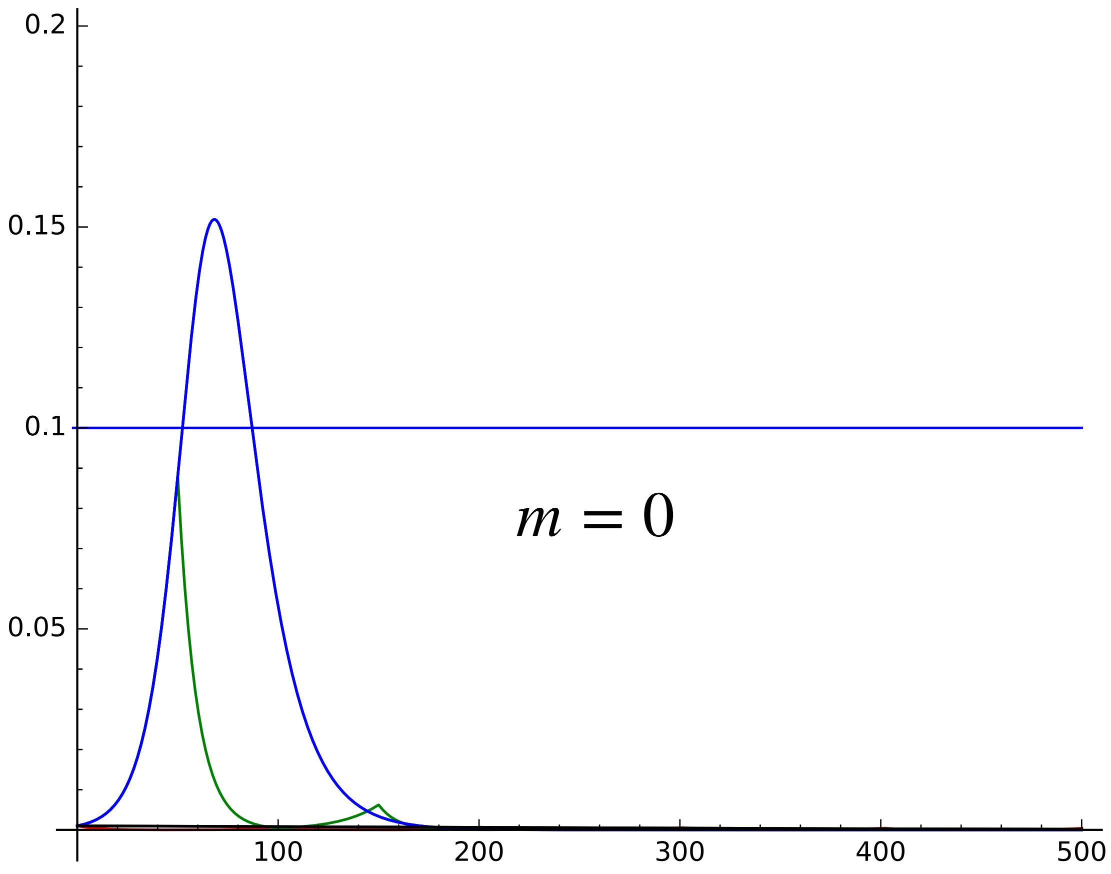


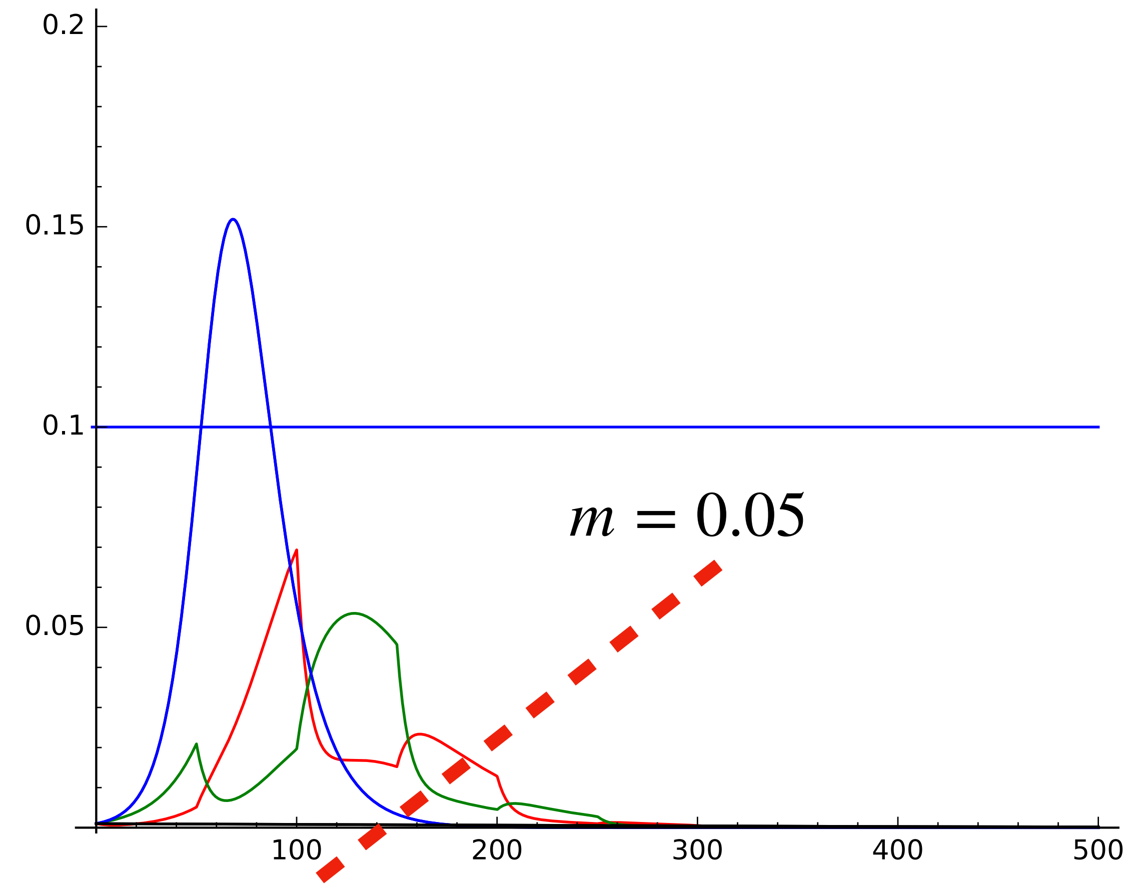


La migration menace le site vertueux

T = 100

Modèle diffusion



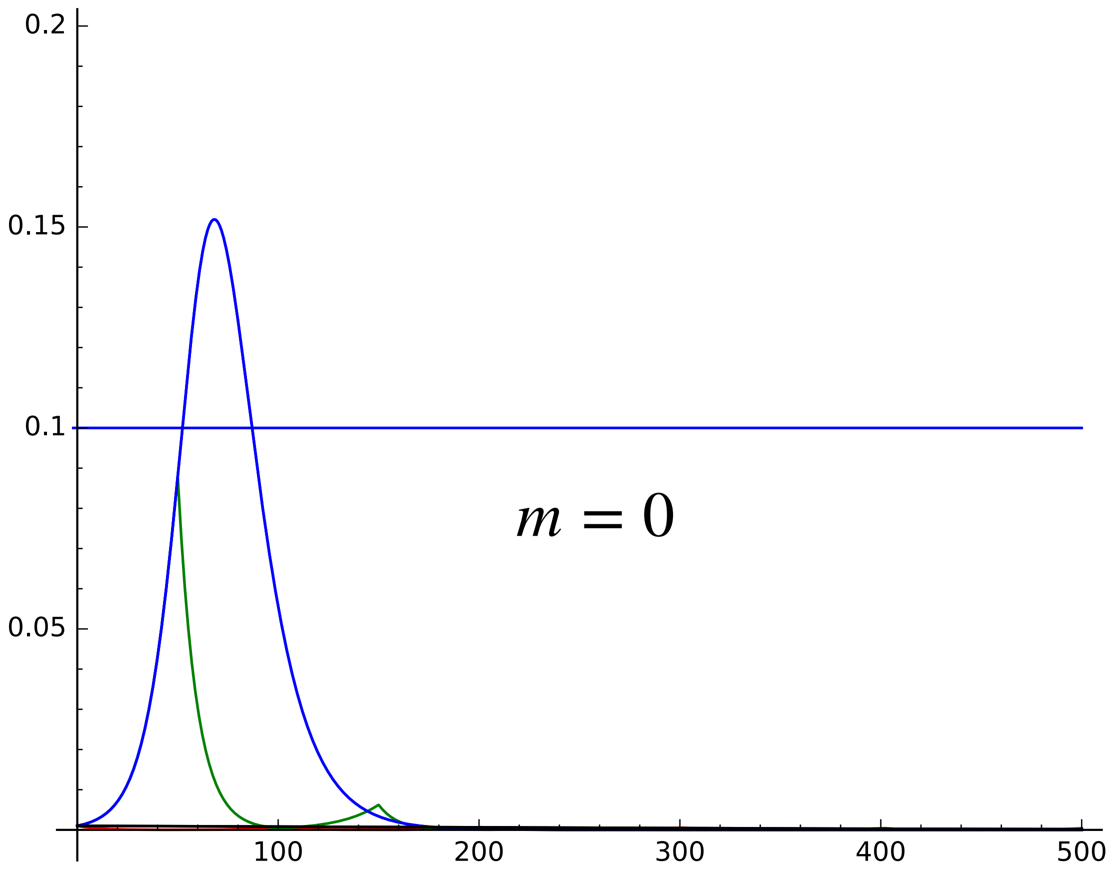


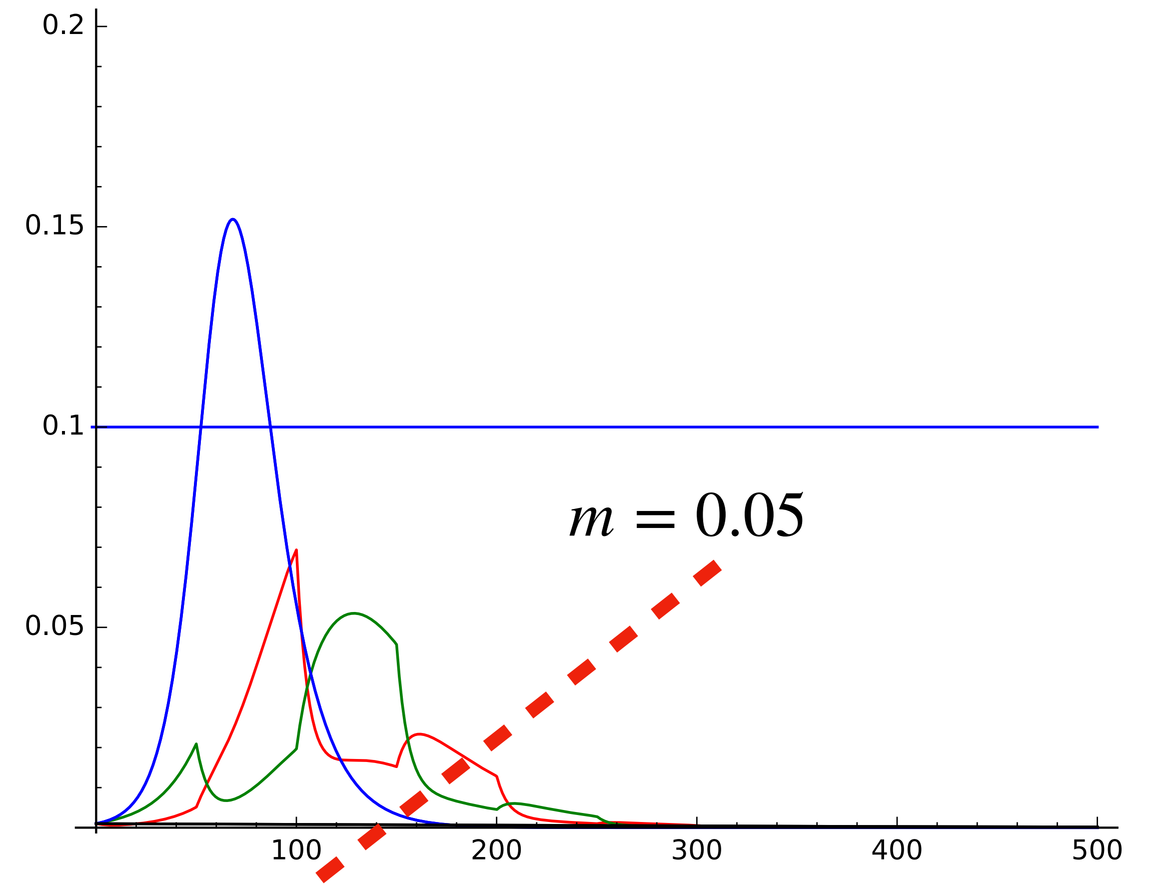
Contrôle des frontières

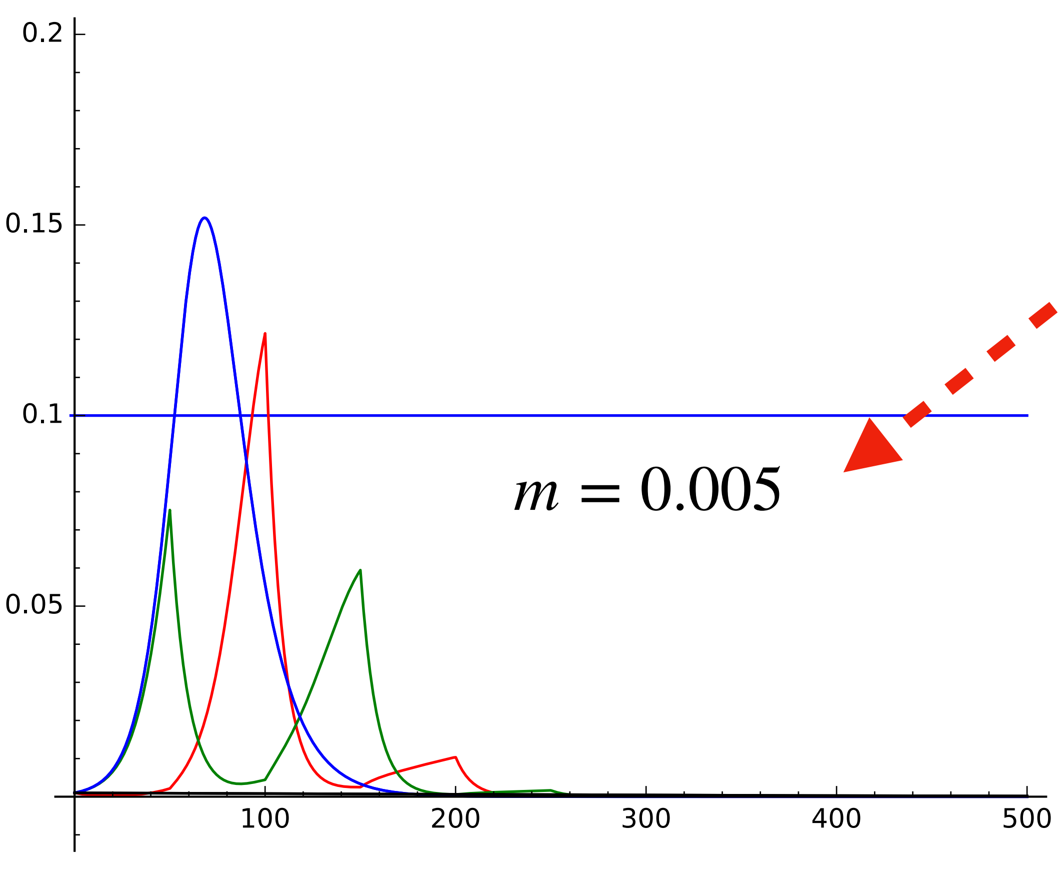
$m = 0.005$

T = 100

Modèle diffusion



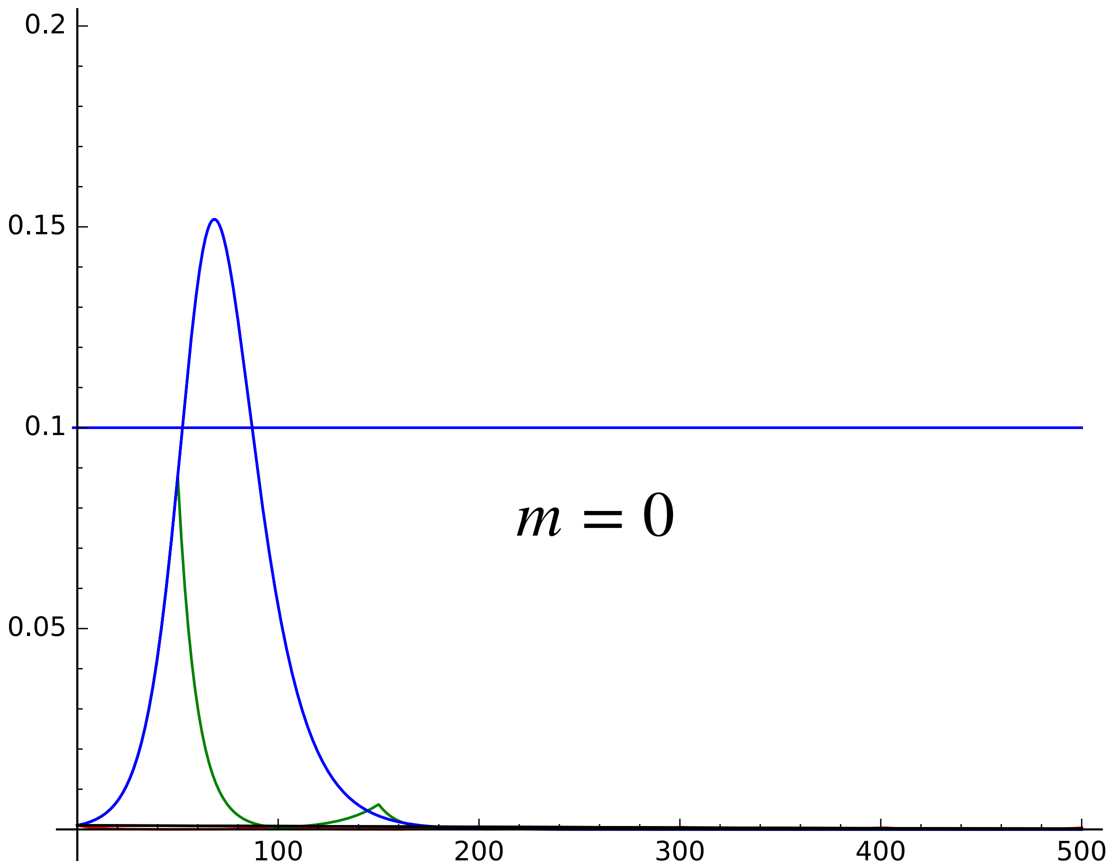


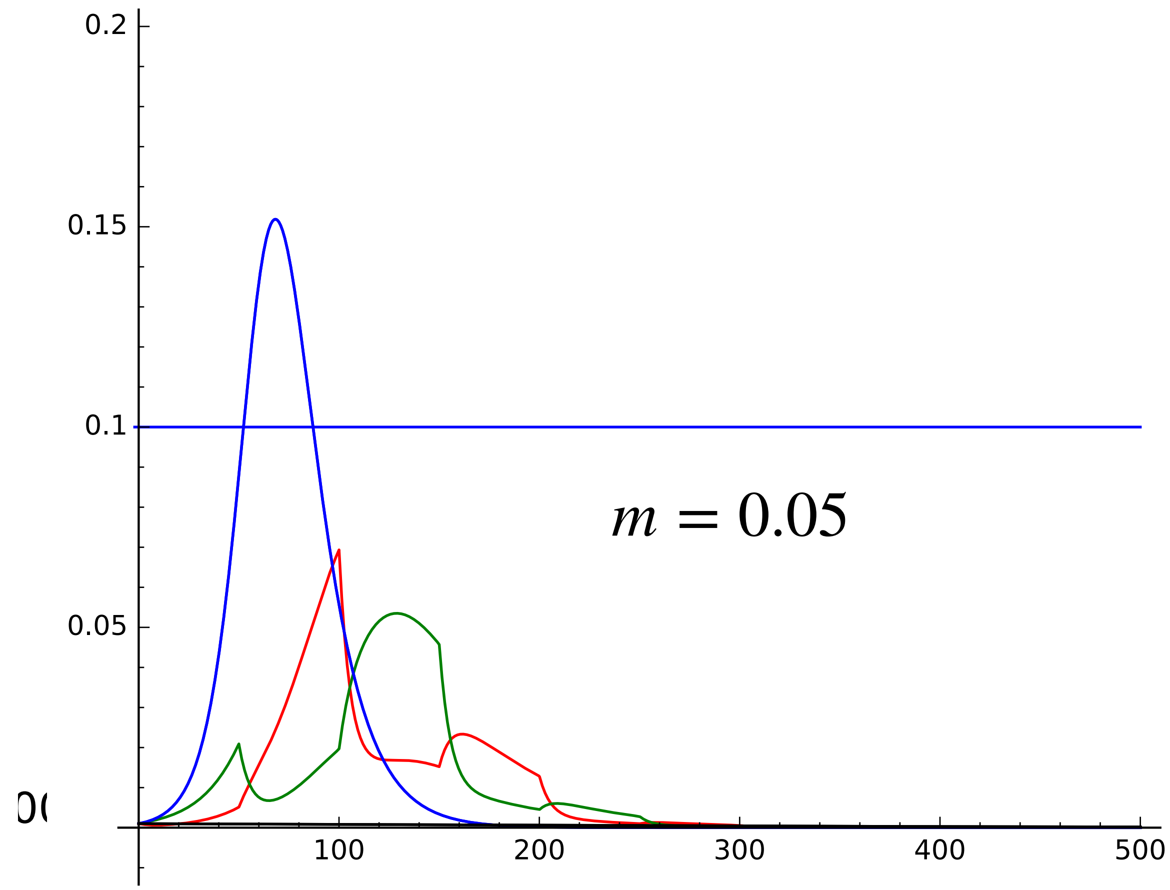


Contrôle des frontières

T = 100

Modèle diffusion





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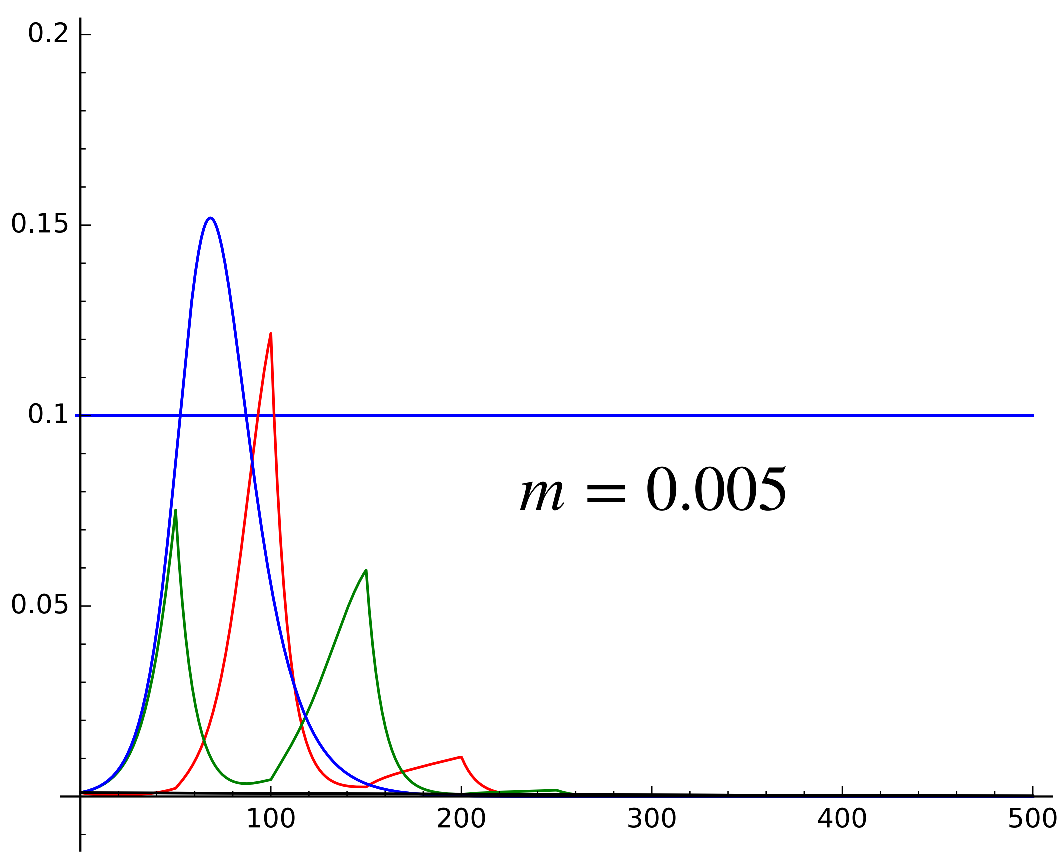
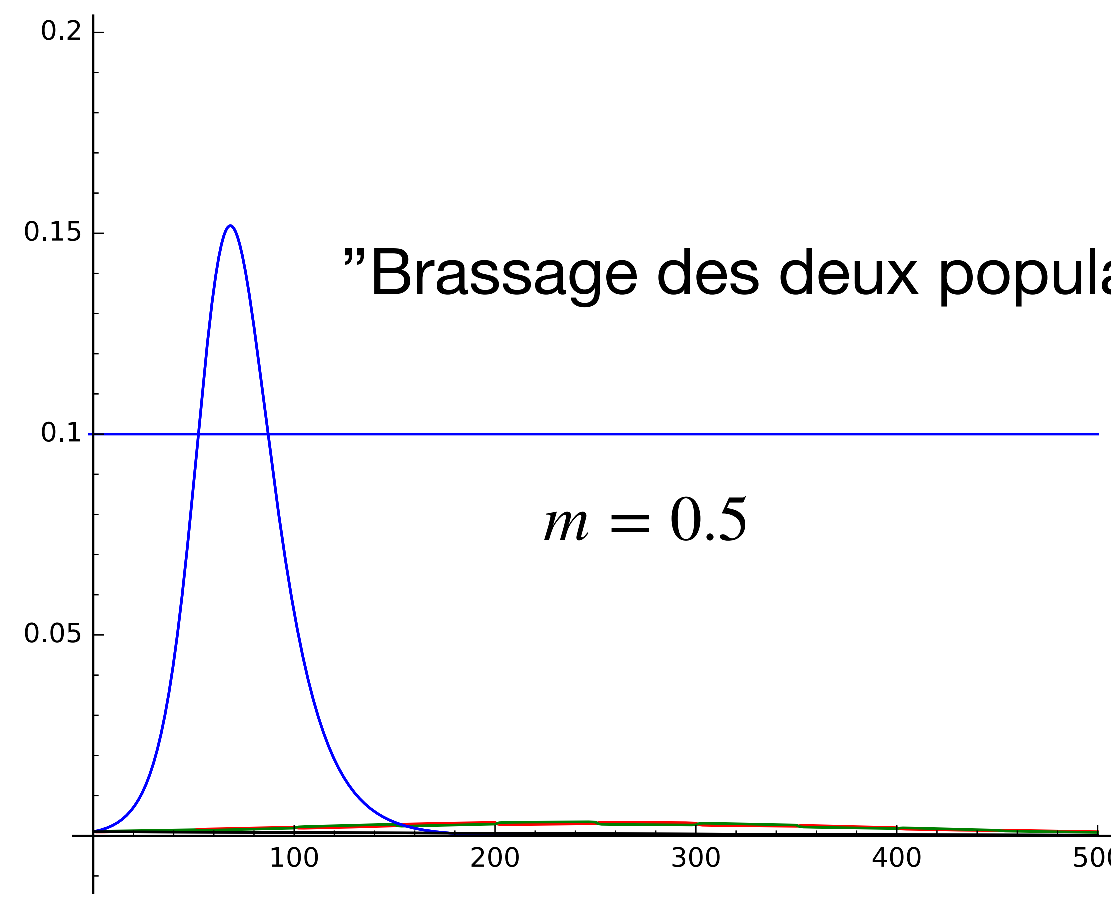




”Brassage des deux populations”

Le Modèle-(±1)

Au démarrage de l'épidémie

$$\xcancel{\frac{dS_i}{dt} = -S_i I_i \beta_i(t) - mS_i + mS_j}$$

$$\frac{dI_i}{dt} = S_i I_i \beta_i(t) - \left[\gamma_i(t) + \mu\right] I_i - mI_i + mI_j \,.$$

$$\xcancel{\frac{dR_i}{dt} = \gamma_i(t) I_i - mR_i + mR_j}$$

$$S_i \approx 1$$

$$\frac{dI_1}{dt} = (1 \times \beta_1 - \gamma_1 - \mu_1) I_1 + m(I_2 - I_1)$$

$$\frac{dI_2}{dt} = (1 \times \beta_2 - \gamma_2 - \mu_2) I_2 + m(I_1 - I_2)$$

$$\frac{dI_1}{dt} = r_1(t) I_1 + m(I_2 - I_1)$$

$$\frac{dI_2}{dt} = r_2(t) I_2 + m(I_1 - I_2)$$

Système linéaire en dimension 2

Analyse (presque) triviale

Work in progress....

M. Benaïm, C. Lobry, T. Sari, E. Strickler

M.B. University of Neuchâtel

C.L. University of Nice

T.S. INRAE Montpellier

E.S. Institut Élie Cartan de Lorraine

= **BLSS**

Le modèle le plus simple qui contient les caractéristiques essentielles de l'**inflation**

$$\Sigma(\varepsilon, m, T) \quad \left\{ \begin{array}{rcl} \dfrac{dx_1}{dt} & = & (+u(t) - \varepsilon)x_1 + m(x_2 - x_1) \\ \dfrac{dx_2}{dt} & = & (-u(t) - \varepsilon)x_2 + m(x_1 - x_2) \end{array} \right.$$

$$t \in [0, T[\Rightarrow u(t) = 1 \qquad t \in [T, 2T[\Rightarrow u(t) = -1$$

Journal of Mathematical Biology (2022) 85:24
https://doi.org/10.1007/s00285-022-01791-7

Mathematical Biology

Dispersal-induced growth in a time-periodic environment

Guy Katriel[1]

Received: 26 November 2021 / Revised: 1 August 2022 / Accepted: 16 August 2022

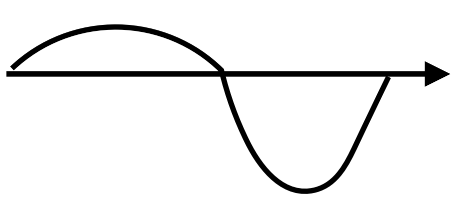
Une remarque (importante) de G. Katriel

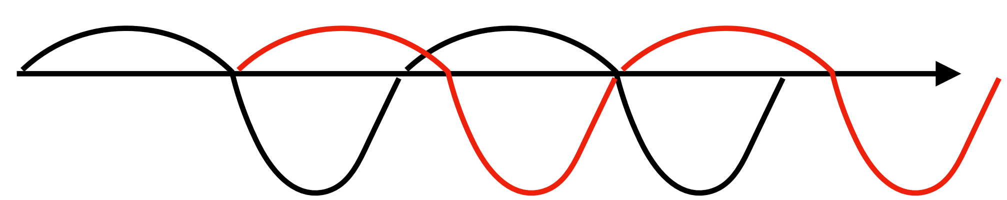
$$\frac{dx}{dt} = r(t)x \quad x(T) = x(0)\,\exp\left(\int_0^T r(s)ds\right)$$

$$\frac{dx_1}{dt} = r_1(t)x_1 + m(x_2 - x_1)$$

$$\frac{dx_2}{dt} = r_2(t)x_2 + m(x_1 - x_2)$$

$$\frac{d(x_1 + x_2)}{dt} \leq \max(r_1(t), r_2(t))(x_1 + x_2)$$





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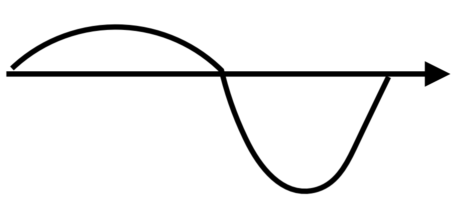
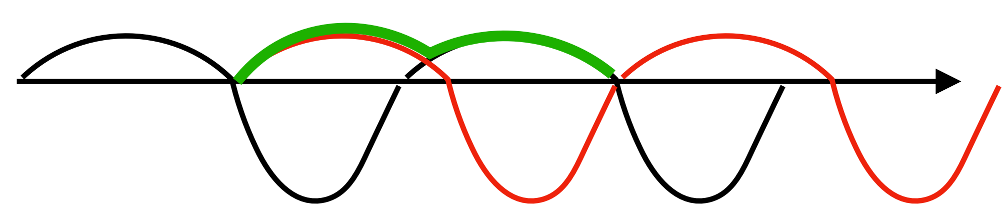
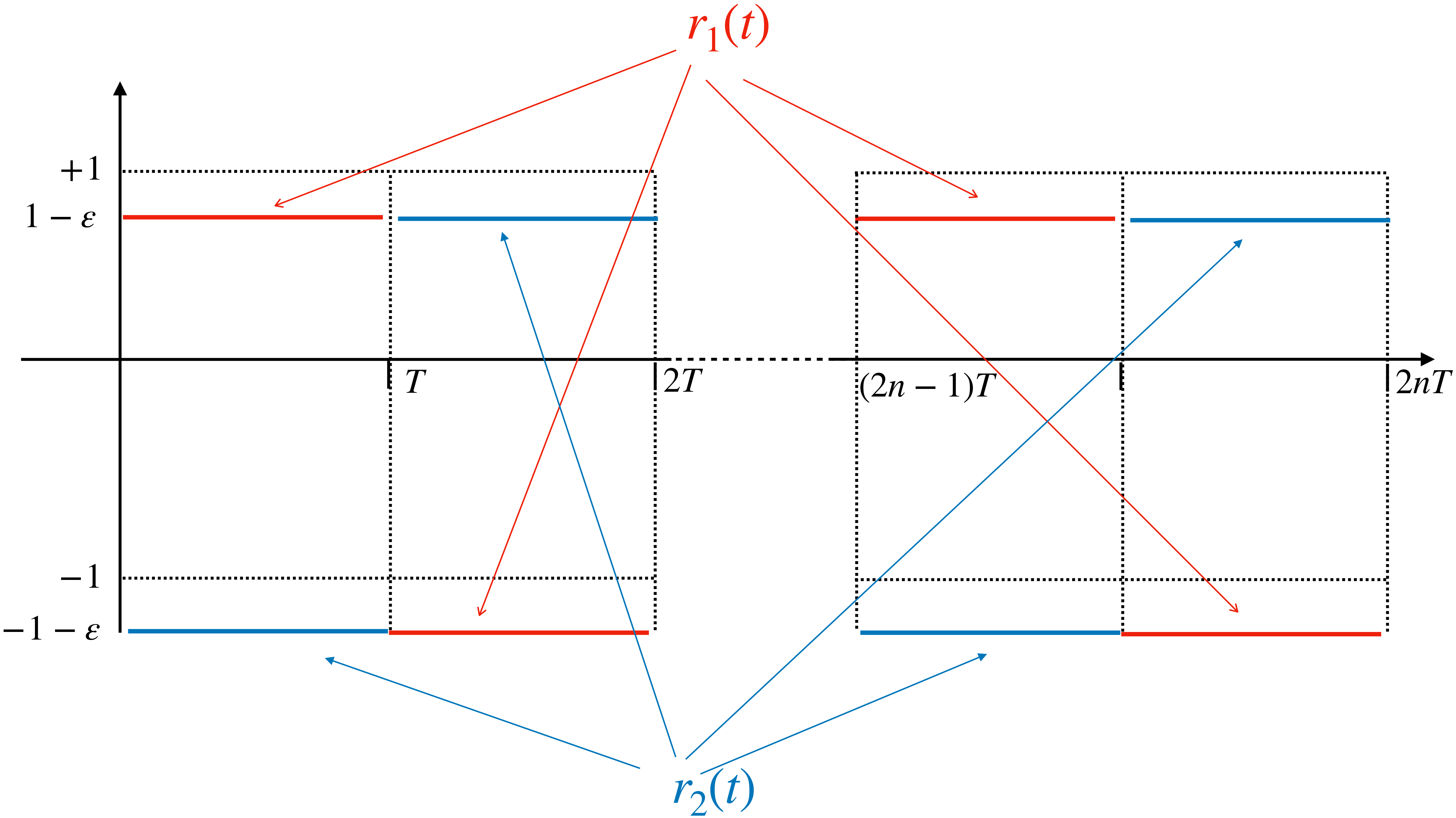




Le modèle (±1)



Les seconds membres discontinus ne posent pas de problèmes

Systèmes commutés

The (±1)model

Forçage périodique constant par morceaux

$$\Sigma(\varepsilon, m, T) \quad \begin{cases} \dfrac{dx_1}{dt} & = \quad (+u(t) - \varepsilon)x_1 + m(x_2 - x_1) \\ \dfrac{dx_2}{dt} & = \quad (-u(t) - \varepsilon)x_2 + m(x_1 - x_2) \end{cases}$$

$$t \in [0, T[\Rightarrow u(t) = 1 \qquad t \in [T, 2T[\Rightarrow u(t) = -1$$

Système périodique de périodes 2T

Considéré comme .....

.... le système commuté

Forçage périodique constant par morceaux

$$\Sigma^+(\varepsilon,m) \quad \left\{\begin{array}{rcl} \dfrac{dx_1}{dt} & = & (+1-\varepsilon)x_1+m(x_2-x_1) \\ \dfrac{dx_2}{dt} & = & (-1-\varepsilon)x_2+m(x_1-x_2) \end{array}\right.$$

sur [0,T[

commute

$$\Sigma^-(\varepsilon,m) \quad \left\{\begin{array}{rcl} \dfrac{dx_1}{dt} & = & (-1-\varepsilon)x_1+m(x_2-x_1) \\ \dfrac{dx_2}{dt} & = & (+1-\varepsilon)x_2+m(x_1-x_2) \end{array}\right.$$

sur [T,2T[

commute

$$\Sigma^+(\varepsilon,m) \quad \left\{\begin{array}{rcl} \dfrac{dx_1}{dt} & = & (+1-\varepsilon)x_1+m(x_2-x_1) \\ \dfrac{dx_2}{dt} & = & (-1-\varepsilon)x_2+m(x_1-x_2) \end{array}\right.$$

sur [2T,3T[

etc…

Une façon d’introduite les PDMP

….les durées successives ne sont pas constantes, mais des v.a. indépendantes

$$\Sigma^{+}(\varepsilon, m) \quad \left\{ \begin{aligned} \frac{dx_1}{dt} &= (+1-\varepsilon)x_1 + m(x_2 - x_1) \\ \frac{dx_2}{dt} &= (-1-\varepsilon)x_2 + m(x_1 - x_2) \end{aligned} \right.$$

~~sur [0,T[~~ T_1

commute

$$\Sigma^{-}(\varepsilon, m) \quad \left\{ \begin{aligned} \frac{dx_1}{dt} &= (-1-\varepsilon)x_1 + m(x_2 - x_1) \\ \frac{dx_2}{dt} &= (+1-\varepsilon)x_2 + m(x_1 - x_2) \end{aligned} \right.$$

~~sur [T,2T[~~ T_2

commute

$$\Sigma^{+}(\varepsilon, m) \quad \left\{ \begin{aligned} \frac{dx_1}{dt} &= (+1-\varepsilon)x_1 + m(x_2 - x_1) \\ \frac{dx_2}{dt} &= (-1-\varepsilon)x_2 + m(x_1 - x_2) \end{aligned} \right.$$

~~sur [2T,3T[~~ T_3

etc…

$T_1, T_2, \cdots, T_n, \cdots$

$$P(T < x) = \frac{1}{\lambda}\int_0^x \mathrm{e}^{-\lambda t}dt$$

$$\mathrm{E}[T] = \lambda$$

....les durées successives ne sont pas constantes, mais des v.a. indépendantes

$$\Sigma^+(\varepsilon,m) \quad \begin{cases} \dfrac{dx_1}{dt} = (+1-\varepsilon)x_1 + m(x_2 - x_1) \\ \dfrac{dx_2}{dt} = (-1-\varepsilon)x_2 + m(x_1 - x_2) \end{cases}$$

~~sur [0,T[~~ T_1

commute

$$\Sigma^-(\varepsilon,m) \quad \begin{cases} \dfrac{dx_1}{dt} = (-1-\varepsilon)x_1 + m(x_2 - x_1) \\ \dfrac{dx_2}{dt} = (+1-\varepsilon)x_2 + m(x_1 - x_2) \end{cases}$$

~~sur [T,2T[~~ T_2

$$\Sigma^+(\varepsilon,m) \quad \begin{cases} \dfrac{dx_1}{dt} = (+1-\varepsilon)x_1 + m(x_2 - x_1) \\ \dfrac{dx_2}{dt} = (-1-\varepsilon)x_2 + m(x_1 - x_2) \end{cases}$$

$T_1, T_2, \cdots, T_n, \cdots$ $\qquad P(T < x) = \dfrac{1}{\lambda}\displaystyle\int_0^x \mathrm{e}^{-\lambda t}dt$ $\qquad \mathrm{E}[T] = \lambda$

$\lambda = 10$

```
1.30918738247732)
1.6896908115807752)
6.409479737184405)
11.47888404120739)
2.449504973595646)
0.7323416871722689)
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1.5408555851557373)
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3.851421482224009)
24.928717042065404)
```

The (±1)model

Forçage périodique constant par morceaux

$$\Sigma(\varepsilon, m, T) \quad \begin{cases} \dfrac{dx_1}{dt} = (+u(t) - \varepsilon)x_1 + m(x_2 - x_1) \\ \dfrac{dx_2}{dt} = (-u(t) - \varepsilon)x_2 + m(x_1 - x_2) \end{cases}$$

$$t \in [0, T[\Rightarrow u(t) = 1 \qquad t \in [T, 2T[\Rightarrow u(t) = -1$$

$$M^{u}_{\varepsilon,m} = \begin{pmatrix} u - m - \varepsilon & +m \\ +m & -u - m - \varepsilon \end{pmatrix}$$

$$\begin{pmatrix} x_1(2T) \\ x_2(2T) \end{pmatrix} = M(\varepsilon, m, T) \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$

$$\left.\begin{matrix} \lambda_1(\epsilon, m, T) \\ \lambda_2(\epsilon, m, T) \end{matrix}\right\} \text{ v.p. de } \quad M(\varepsilon, m, T) = \mathrm{e}^{T M^{-1}_{\varepsilon,m}} \mathrm{e}^{T M^{+1}_{\varepsilon,m}}$$

En principe on peut résoudre à la main

$$M^{u}_{\varepsilon,m} = \begin{pmatrix} u-m-\varepsilon & +m \\ +m & -u-m-\varepsilon \end{pmatrix}$$

→ calculer les V. P.

$$M(\varepsilon,m,T) = \mathrm{e}^{TM^{-1}_{\varepsilon,m}}\mathrm{e}^{TM^{+1}_{\varepsilon,m}}$$

Calculer le produit

et calculer les v.p.

Forçage périodique constant par morceaux

Maple calcule

$$\lambda_1(\varepsilon, m, T) = \cdots$$

$$\frac{1}{2\left(\mathrm{e}^{TA}\right)^2\left(m^2+1\right)\left(\mathrm{e}^{Tm}\right)^2\left(\mathrm{e}^{T\varepsilon}\right)^2}\left(\left(\mathrm{e}^{TA}\right)^4 m^2+2\left(\mathrm{e}^{TA}\right)^2+m^2+\sqrt{\left(\mathrm{e}^{TA}\right)^8 m^4+4\left(\mathrm{e}^{TA}\right)^6 m^2-2\left(\mathrm{e}^{TA}\right)^4 m^4-8\left(\mathrm{e}^{TA}\right)^4 m^2+4\left(\mathrm{e}^{TA}\right)^2 m^2+m^4}\right)$$

$$\lambda_2(\varepsilon, m, T) = \cdots$$

$$\frac{1}{2\left(\mathrm{e}^{TA}\right)^2\left(m^2+1\right)\left(\mathrm{e}^{Tm}\right)^2\left(\mathrm{e}^{T\varepsilon}\right)^2}\left(\left(\mathrm{e}^{TA}\right)^4 m^2+2\left(\mathrm{e}^{TA}\right)^2+m^2-\sqrt{\left(\mathrm{e}^{TA}\right)^8 m^4+4\left(\mathrm{e}^{TA}\right)^6 m^2-2\left(\mathrm{e}^{TA}\right)^4 m^4-8\left(\mathrm{e}^{TA}\right)^4 m^2+4\left(\mathrm{e}^{TA}\right)^2 m^2+m^4}\right)$$

$$\text{with } A = \sqrt{m^2+1}.$$

pas particulièrement éclairant

$$\lambda_1(\varepsilon, m, T) = \cdots$$

$$\frac{1}{2\left(e^{TA}\right)^2\left(m^2+1\right)\left(e^{Tm}\right)^2\left(e^{T\varepsilon}\right)^2}\left(\left(e^{TA}\right)^4 m^2 + 2\left(e^{TA}\right)^2 + m^2 + \sqrt{\left(e^{TA}\right)^8 m^4 + 4\left(e^{TA}\right)^6 m^2 - 2\left(e^{TA}\right)^4 m^4 - 8\left(e^{TA}\right)^4 m^2 + 4\left(e^{TA}\right)^2 m^2 + m^4}\right)$$

$$\lambda_2(\varepsilon, m, T) = \cdots$$

$$\frac{1}{2\left(e^{TA}\right)^2\left(m^2+1\right)\left(e^{Tm}\right)^2\left(e^{T\varepsilon}\right)^2}\left(\left(e^{TA}\right)^4 m^2 + 2\left(e^{TA}\right)^2 + m^2 - \sqrt{\left(e^{TA}\right)^8 m^4 + 4\left(e^{TA}\right)^6 m^2 - 2\left(e^{TA}\right)^4 m^4 - 8\left(e^{TA}\right)^4 m^2 + 4\left(e^{TA}\right)^2 m^2 + m^4}\right)$$

with $A = \sqrt{m^2+1}$.

On demande à Maple le graphe de l'exposant de Liapunov

$$(m, T) \mapsto \frac{\ln(\lambda_1(\varepsilon, m, T))}{2T} = \Delta(\varepsilon, m, T)$$

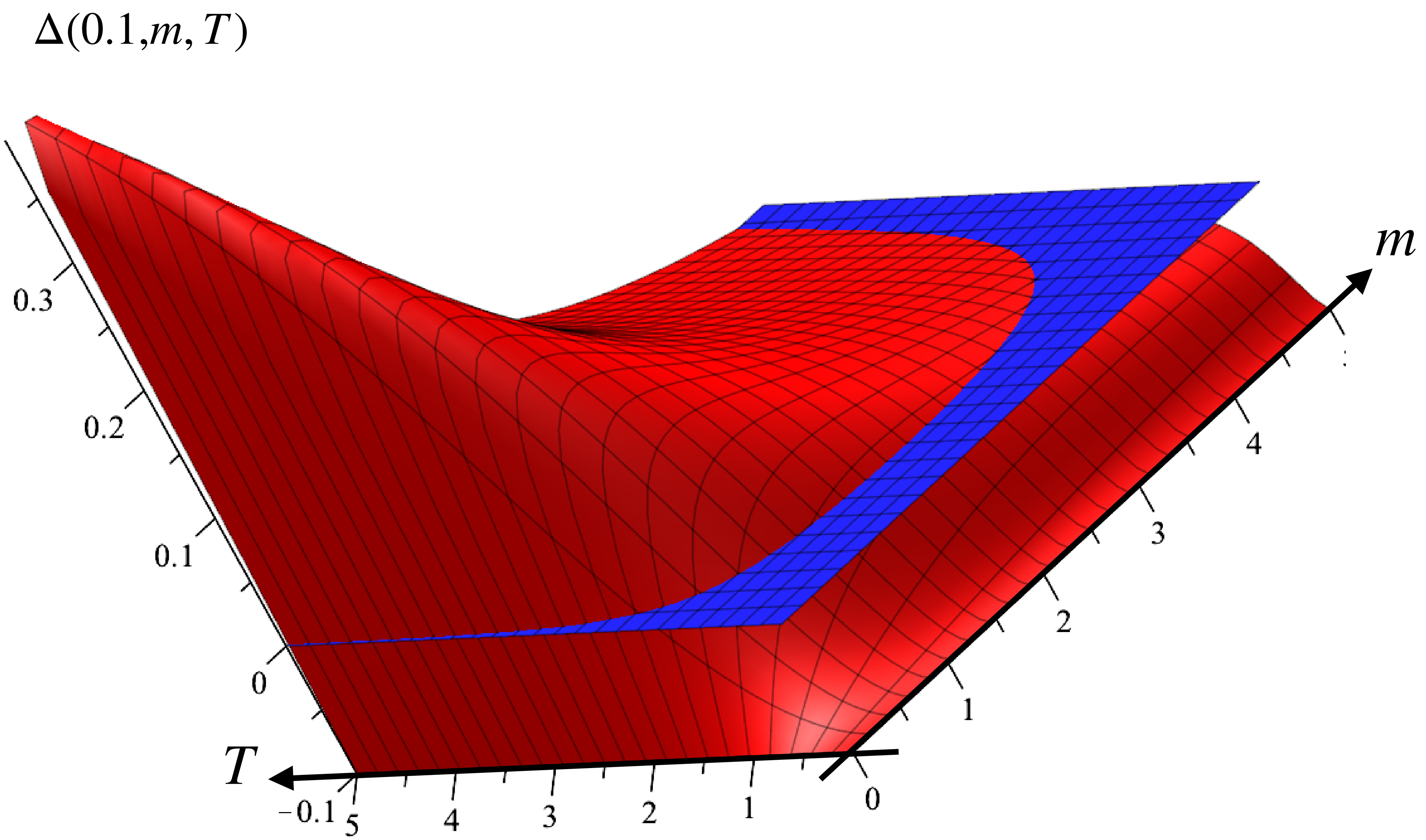
puisque

$$\lim_{t\to\infty} \frac{1}{t}\ln(x_1(t)) = \lim_{t\to\infty} \frac{1}{t}\ln(x_2(t)) = \Delta(\varepsilon, m, T)$$

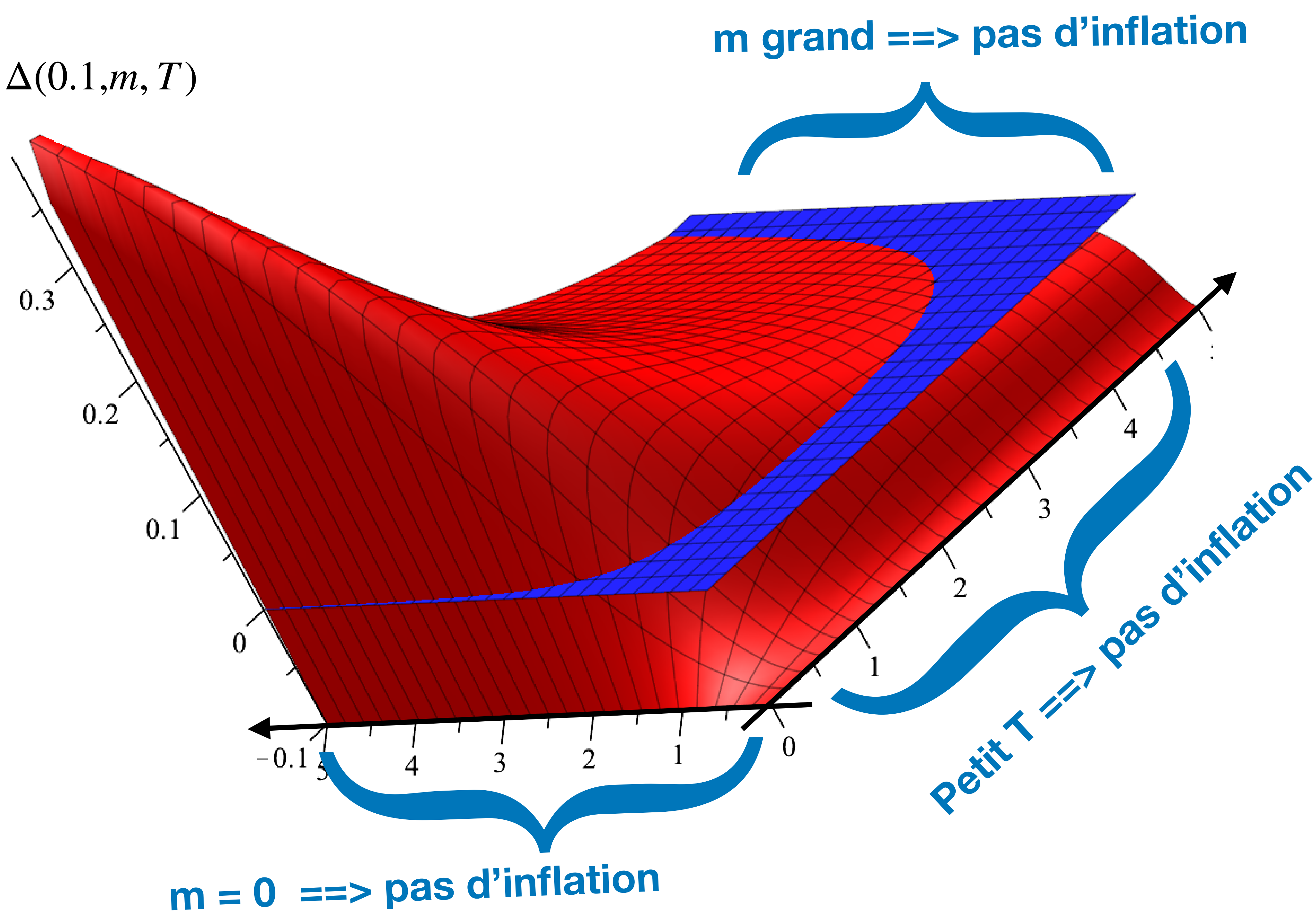
DIG

On dit qu'il y a inflation si :

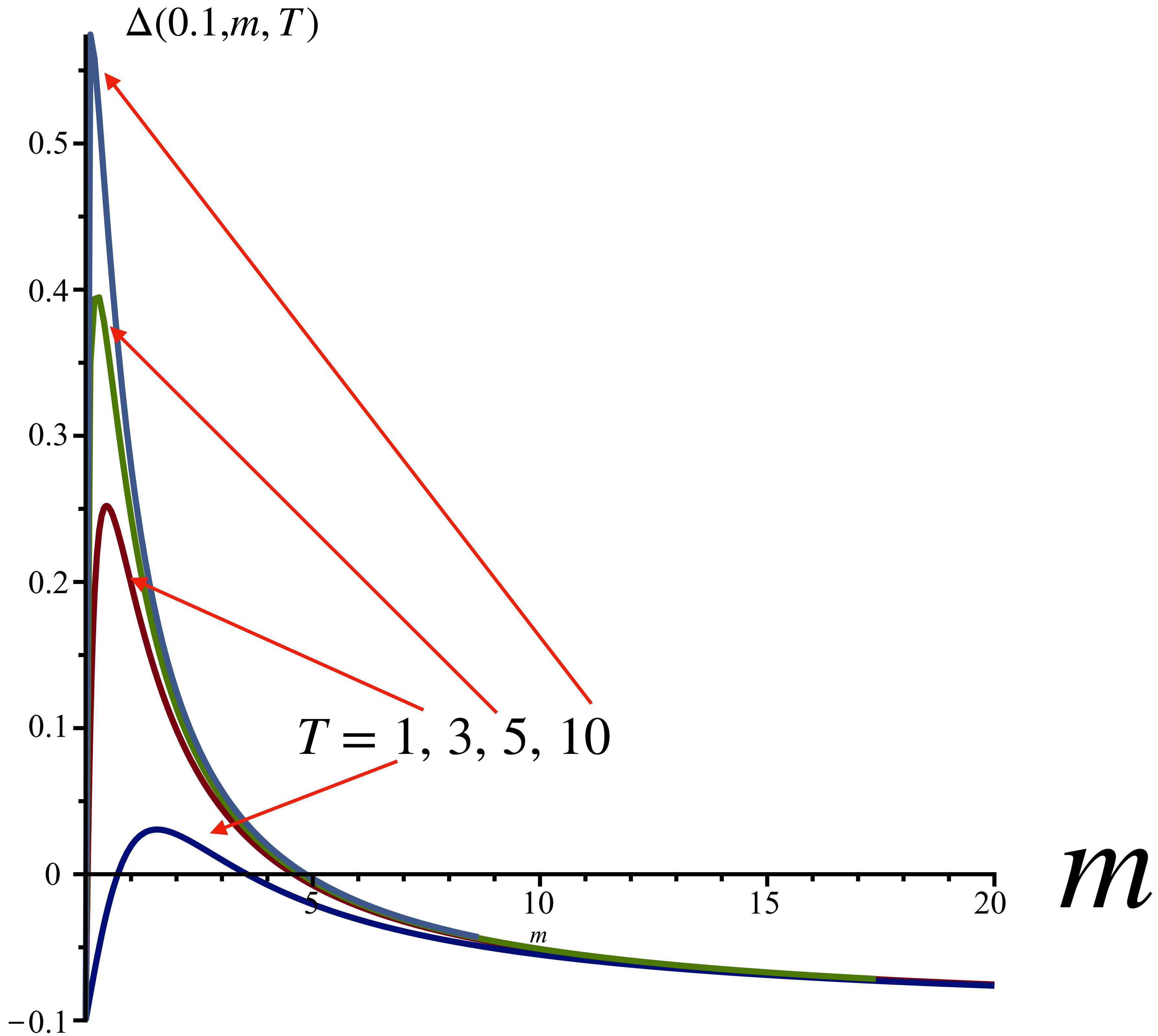
$$\Delta(\varepsilon, m, T) > 0$$



Grands T ==> Forte sensibilité à m



Forçage périodique constant par morceaux



. m = 0 ==> pas d'inflation

Trivial :Les deux sites décroissent indépendamment

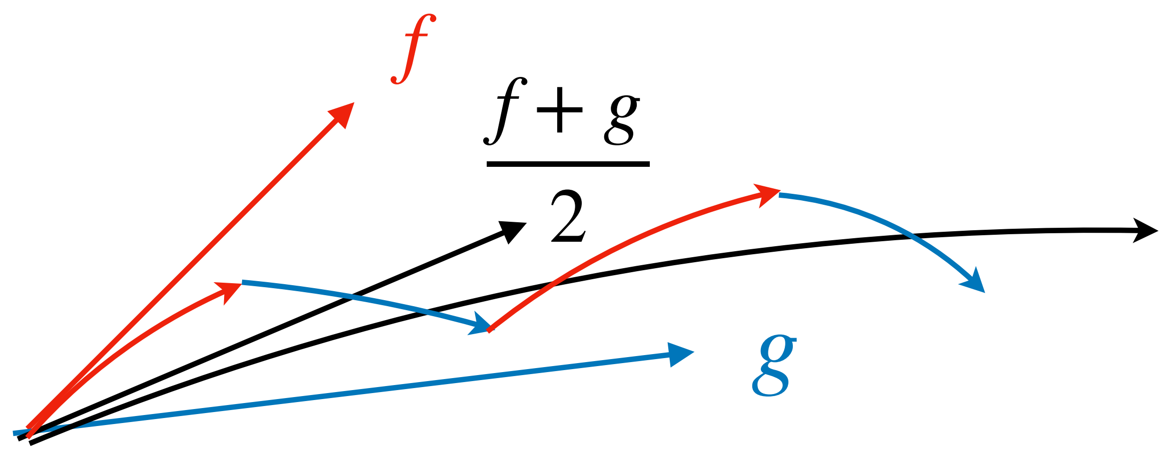
$$\left.\begin{aligned} x_1(2T) &= \exp((T(-1-\varepsilon))\exp(T(1-\varepsilon))x_1(0) = \exp(-2T\varepsilon)x_1(0) \\ x_2(2T) &= \exp((T(-1-\varepsilon))\exp(T(1-\varepsilon))x_2(0) = \exp(-2T\varepsilon)x_2(0) \end{aligned}\right\} \Longrightarrow \frac{\ln(\lambda_1(\varepsilon, m, T))}{2T} = -\,\varepsilon$$

T petit ==> pas d'inflation

Forçage périodique constant par morceaux

Presque trivial: voir la figure:

En général



Moyennisation

Quand T —> 0
les solutions du commuté convergent vers la solution de la 1/2 somme

Dans notre cas

$$\Sigma^+(\varepsilon, m) \quad \begin{cases} \dfrac{dx_1}{dt} = (+1-\varepsilon)x_1 + m(x_2 - x_1) \\ \dfrac{dx_2}{dt} = (-1-\varepsilon)x_2 + m(x_1 - x_2) \end{cases}$$

sur [0,T[

commute

$$\Sigma^-(\varepsilon, m) \quad \begin{cases} \dfrac{dx_1}{dt} = (-1-\varepsilon)x_1 + m(x_2 - x_1) \\ \dfrac{dx_2}{dt} = (+1-\varepsilon)x_2 + m(x_1 - x_2) \end{cases}$$

sur [T,2T[

$$\frac{\Sigma^+ + \Sigma^-}{2} \quad \begin{cases} \dfrac{dx_1}{dt} = -(\varepsilon + m)x_1 + m x_2 \\ \dfrac{dx_2}{dt} = m x_1 - (\varepsilon + m)x_2 \end{cases} \Longrightarrow \frac{\lambda_1(\varepsilon, m, T)}{2T} = -\varepsilon$$

Forçage périodique constant par morceaux

. m —> $+\infty$ ==> pas d'inflation

Intuitivement

$$\frac{dx_1}{dt} = f(x_1, x_2) + m(x_2 - x_1) \qquad u = (x_1 + x_2)/2 \qquad x_1 = (u + v)$$

$$\frac{dx_2}{dt} = g(x_1, x_2) + m(x_1 - x_2) \qquad v = (x_1 - x_2)/2 \qquad x_2 = (u - v)$$

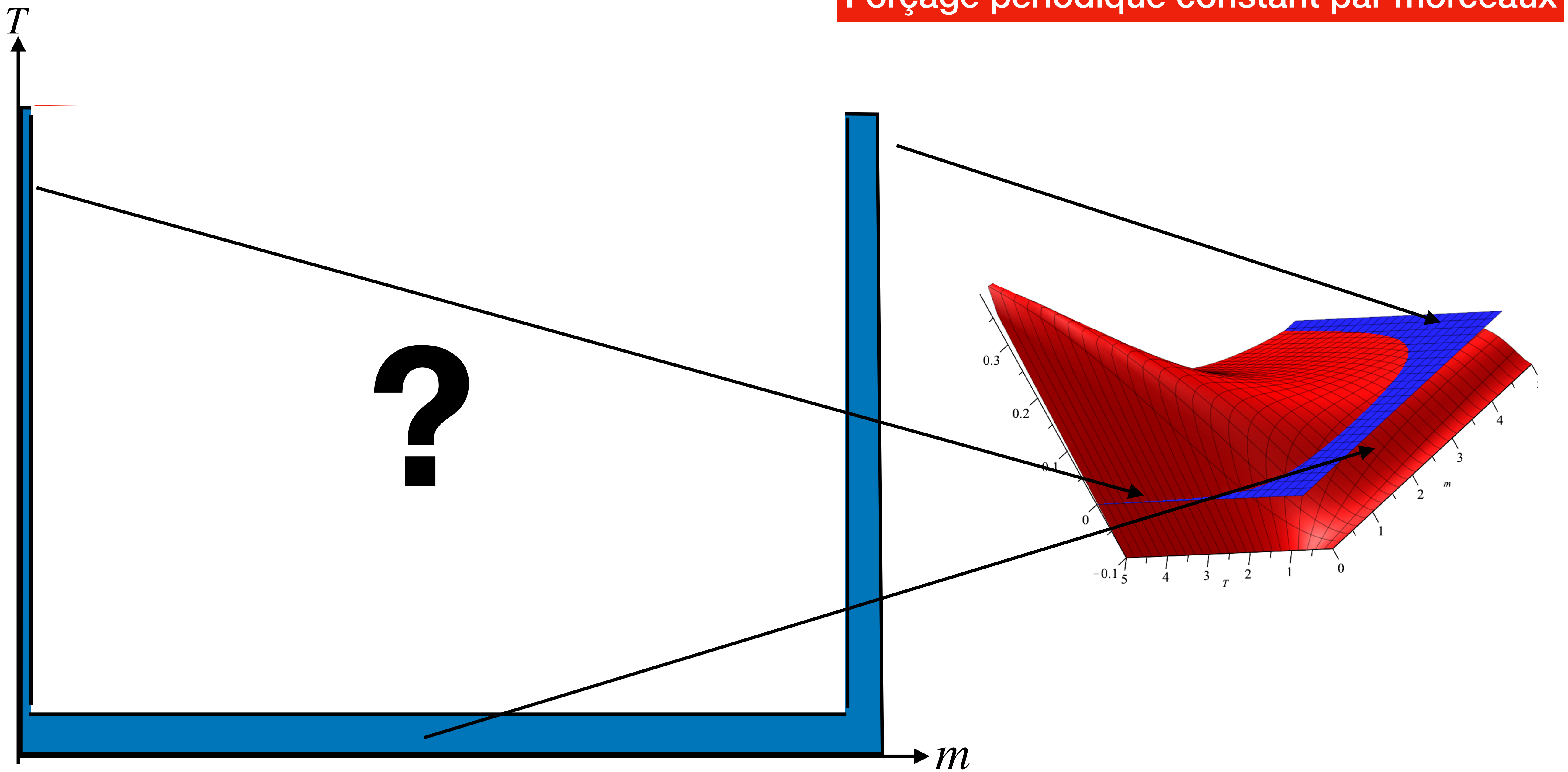
$$\frac{du}{dt} = \frac{1}{2}(f(u, v) + g(u, v))$$

$$\frac{dv}{dt} = (f - g)/2 - mv$$

Théorème de Tychonov

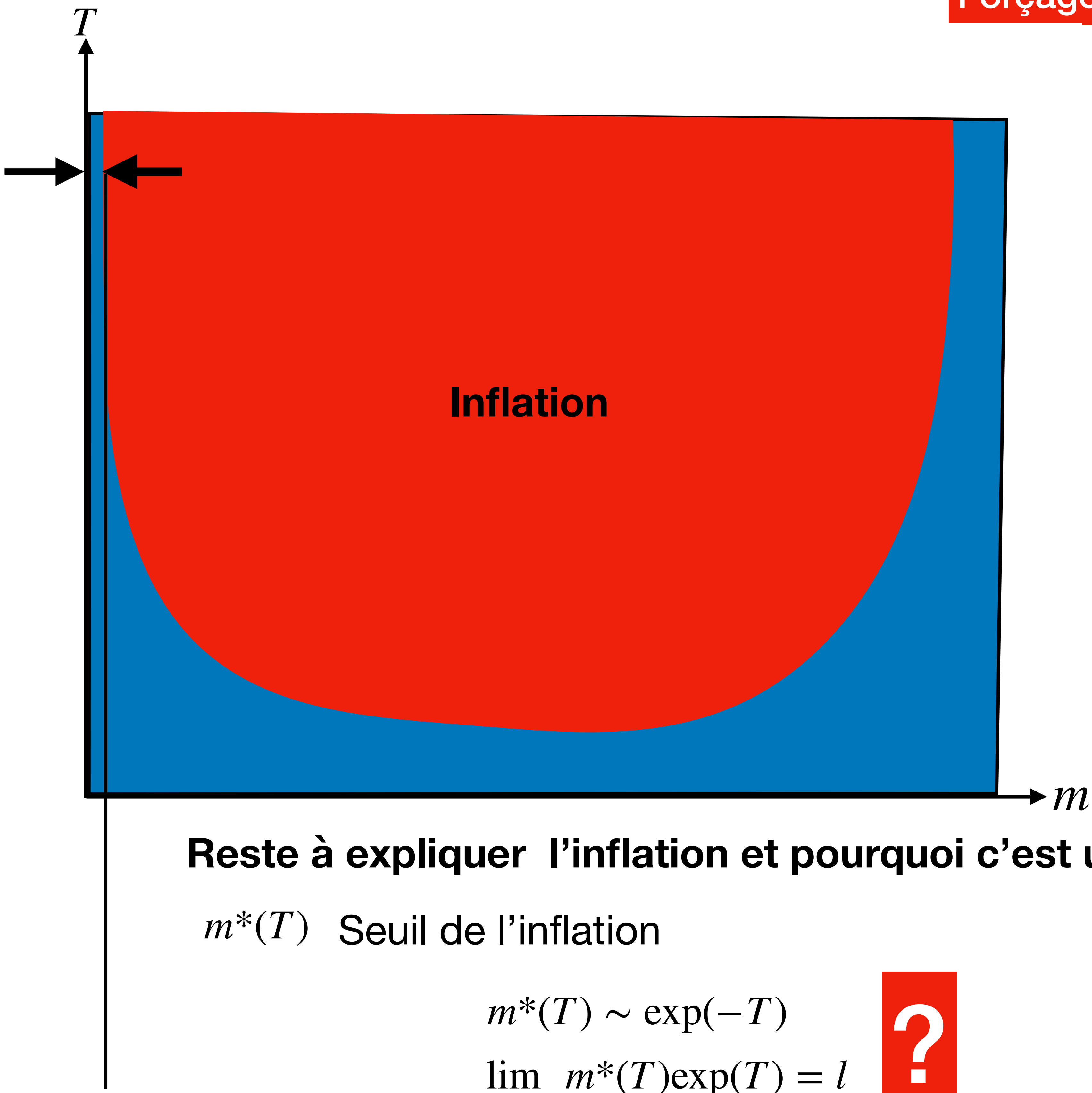
$$mi\,.\,g\,. \Longrightarrow v \approx 0$$

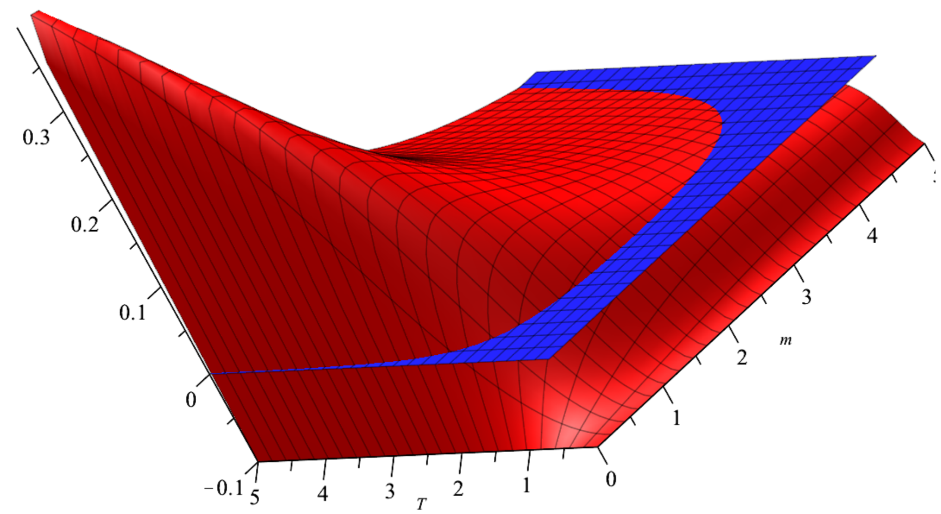
Forçage périodique constant par morceaux



On a expliqué ce qui se passe au bord avec des arguments généraux, non basés sur la forme explicite du modèle (±1)

Forçage périodique constant par morceaux





Reste à expliquer l'inflation et pourquoi c'est un phénomène brutal.

$m^*(T)$ Seuil de l'inflation

$$m^*(T) \sim \exp(-T)$$

$$\lim_{T \to +\infty} m^*(T)\exp(T) = l$$

?

Se démontre par un calcul sur la formule explicite :

$$\Delta(\varepsilon,m,T);=2\ln\frac{m^2b^4+2b^2+m^2+m(b^2-1)\sqrt{C}}{2(1+m^2)b^2}-4(m+\varepsilon)T$$

avec $b=e^{T\sqrt{1+m^2}}$ et $C=m^2b^4+2m^2b^2+4b^2+m^2.$

…fait ici

$$\Delta(\varepsilon,m,T);=2\ln\frac{m^2b^4+2b^2+m^2+m(b^2-1)\sqrt{C}}{2(1+m^2)b^2}-4(m+\varepsilon)T \tag{72}$$

$$\text{with } b=e^{T\sqrt{1+m^2}} \text{ and } C=m^2b^4+2m^2b^2+4b^2+m^2. \tag{73}$$

Equation (72) is equivalent to :

$$\frac{m^2b^4+2b^2+m^2+m(b^2-1)\sqrt{C}}{2(1+m^2)b^2}=\mathrm{e}^{2(m+\varepsilon)T} \tag{74}$$

From (73) one have :

$$mb=\mathrm{e}^{Tx}\mathrm{e}^{T\sqrt{1+\mathrm{e}2Tx}}=\mathrm{e}^{T(1+x+\frac{1}{2}\mathrm{e}^{2Tx}(1+\mathrm{o}))} \tag{75}$$

since for $x<0$ we have $T\mathrm{e}^{2Tx}=\mathrm{o}$ we deduce $mb=\mathrm{e}^{T(1+x+\mathrm{o})}$ which tends to ∞ as long as $x>-1$. from which we deduce that as long as $x>-1$:

$$m^2b^4+2b^2+m^2=m^2b^4(1+\mathrm{o}) \qquad m(b^2-)\sqrt{C}=m^2b^4(1+\mathrm{o}) \tag{76}$$

which introduced in(73) gives :

$$m^2b^2(1+\mathrm{o})=\mathrm{e}^{2T(1+x)}(1+\mathrm{o})=\mathrm{e}^{2T(\varepsilon+\mathrm{o})} \tag{77}$$

from which we deduce :

$$2T(1+x)(1+\mathrm{o})=2T(\varepsilon+\mathrm{o})\Longrightarrow x=-(1-\varepsilon)+\mathrm{o} \tag{78}$$

which is the evaluation of proposition 2.4.

$$\Delta(\varepsilon,m,T) := 2\ln\frac{m^2b^4+2b^2+m^2+m(b^2-1)\sqrt{C}}{2(1+m^2)b^2} - 4(m+\varepsilon)T \tag{72}$$

with $b = e^{T\sqrt{1+m^2}}$ and $C = m^2b^4+2m^2b^2+4b^2+m^2$. (73)

Equation (72) is equivalent to :

$$\frac{m^2b^4+2b^2+m^2+m(b^2-1)\sqrt{C}}{2(1+m^2)b^2} = \mathrm{e}^{2(m+\varepsilon)T} \tag{74}$$

From (73) one have :

$$mb = \mathrm{e}^{Tx}\mathrm{e}^{T\sqrt{1+m^2}} = \mathrm{e}^{T(1+x+\frac{1}{2}\mathrm{e}^{2Tx}(1+\mathrm{o}))} \tag{75}$$

since for $x<0$ we have $T\mathrm{e}^{2Tx} = \mathrm{o}$ we deduce $mb = \mathrm{e}^{T(1+x+\mathrm{o})}$ which tends to ∞ as long as $x > -1$. from which we deduce that as long as $x > -1$:

$$m^2b^4+2b^2+m^2 = m^2b^4(1+\mathrm{o}) \qquad m(b^2-1)\sqrt{C} = m^2b^4(1+\mathrm{o}) \tag{76}$$

La suite donnera une présentation plus conceptuelle

which introduced in(73) gives :

$$m^2b^2(1+\mathrm{o}) = \mathrm{e}^{2T(1+x)}(1+\mathrm{o}) = \mathrm{e}^{2T(\varepsilon+\mathrm{o})} \tag{77}$$

from which we deduce :

$$2T(1+x)(1+\mathrm{o}) = 2T(\varepsilon+\mathrm{o}) \Longrightarrow x = -(1-\varepsilon)+\mathrm{o} \tag{78}$$

which is the evaluation of proposition 2.4.

Nouvelles variables

Forçage périodique constant par morceaux

$$\frac{dx_1}{dt} = (+u(t)-\varepsilon)x_1 + m(x_2 - x_1)$$
$$\frac{dx_2}{dt} = (-u(t)-\varepsilon)x_2 + m(x_1 - x_2)$$

$$\xi_1 = \ln(x_1) \qquad \xi_2 = \ln(x_2)$$

$$\frac{d\xi_1}{dt} = (+u(t)-\varepsilon) + m\left(e^{\xi_2-\xi_1} - 1\right)$$
$$\frac{d\xi_2}{dt} = (-u(t)-\varepsilon) + m\left(e^{\xi_1-\xi_2} - 1\right)$$

Nouvelles variables

Forçage périodique constant par morceaux

$$\frac{d\xi_1}{dt} = (+u(t) - \varepsilon) + m\left(e^{\xi_2 - \xi_1} - 1\right)$$

$$\frac{d\xi_2}{dt} = (-u(t) - \varepsilon) + m\left(e^{\xi_1 - \xi_2} - 1\right)$$

$U = \xi_1 + \xi_2$ $\quad V = \xi_1 - \xi_2$

$$\boxed{\begin{aligned} U &= \ln(x_1 \cdot x_2) \\ V &= \ln(x_1 / x_2) \end{aligned}}$$

$$\frac{dU}{dt} = 2\left(m\cosh(V) - m - \varepsilon\right)$$

$$\frac{dV}{dt} = 2(u(t) - m\sinh(V))$$

$$\overline{\Delta}(\varepsilon, mT) = \lim_{t\to\infty}\frac{1}{t}U(t) = \lim_{t\to\infty}\frac{1}{t}\ln(x_1(t)) + \lim_{t\to\infty}\frac{1}{t}\ln(x_2(t)) = 2\Delta(\varepsilon, mT)$$

Nouvelles variables

Forçage périodique constant par morceaux

$$\begin{cases} \dfrac{dU}{dt} & = & 2\,(m\cosh(V) - m - \varepsilon) \\ \dfrac{dV}{dt} & = & 2(u(t) - m\sinh(V)) \end{cases}$$

$F(m,T)$ $$\frac{dV}{dt} = 2(u(t) - m\sinh(V))$$ système commuté 1D

$$U(t) \quad = \quad U_0 + \int_0^t 2\,(m\cosh(V(s) - m - \varepsilon)\,ds$$ Simple quadrature

$$\overline{\Delta}(\varepsilon, mT) = \lim_{t\to\infty} \frac{1}{t} U(t)$$

Nouvelles variables

Forçage périodique constant par morceaux

$$\begin{cases} \dfrac{dU}{dt} & = & 2(m\cosh(V) - m - \varepsilon) \\ \dfrac{dV}{dt} & = & 2(u(t) - m\sinh(V)) \end{cases}$$

$F(m,T)$ $$\frac{dV}{dt} = 2(u(t) - m\sinh(V))$$ Système 1D

$$U(t) = U_0 + \int_0^t 2(m\cosh(V(s) - m - \varepsilon)\, ds$$ Quadrature

$$\overline{\Delta}(\varepsilon, mT) = \lim_{t\to\infty} \frac{1}{t} U(t) = \lim_{t\to\infty} \frac{1}{t} \ln(x_1(t)) + \lim_{t\to\infty} \frac{1}{t} \ln(x_2(t)) = 2\Delta(\varepsilon, mT)$$

Nouvelles variables

Forçage périodique constant par morceaux

$$F(m,T) \qquad \frac{dV}{dt} = 2(u(t) - m\sinh(V))$$

Système commuté 1D

$$F_m^{+1} \quad \frac{dV}{dt} = 2(+1 - m\sinh(V)) \qquad V_m^+ = \sinh^{-1}(+1/m)$$

G.A.S

$$F_m^{-1} \quad \frac{dV}{dt} = 2(-1 - m\sinh(V)) \qquad V_m^- = \sinh^{-1}(-1/m)$$

G.A.S

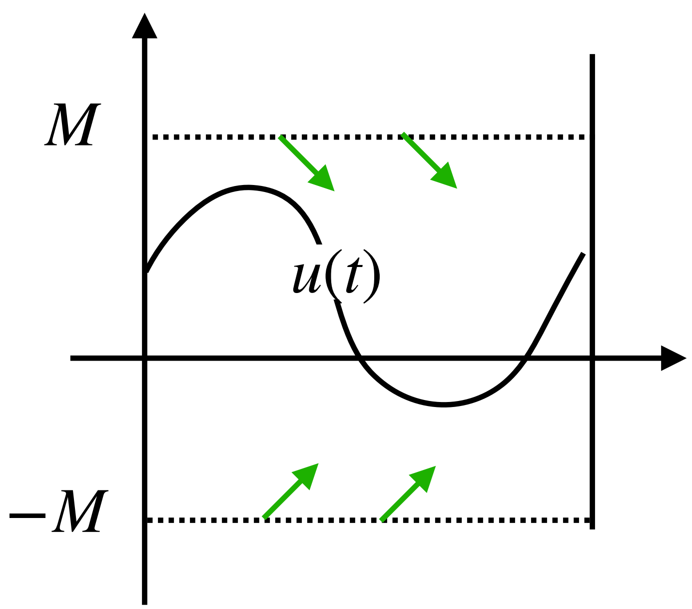
Theorem : F(m,T) a une solution périodique unique .

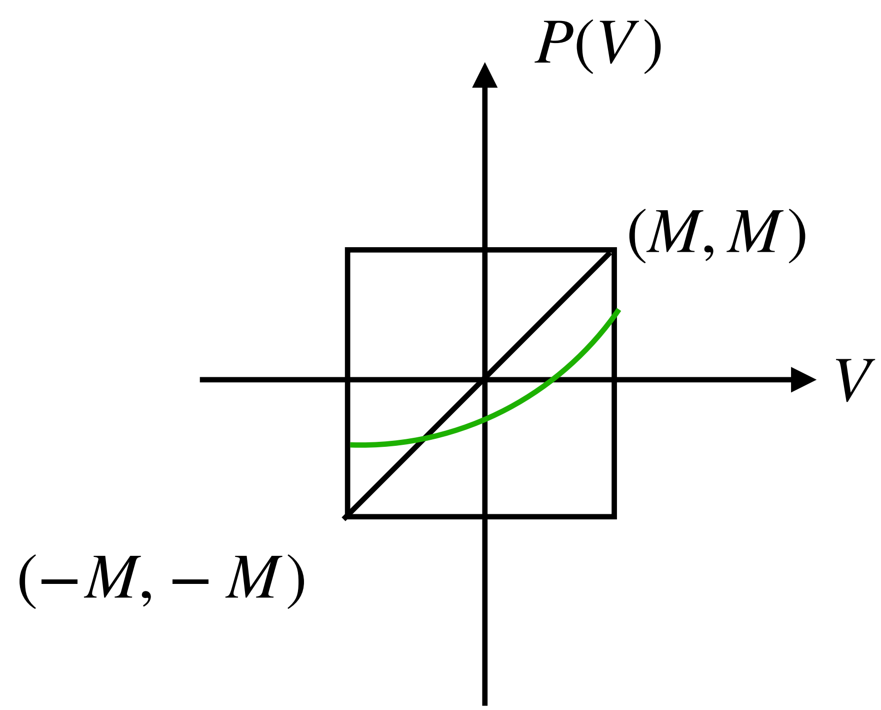
Preuve

$$\boxed{\frac{dV}{dt} = 2(u(t) - m\sinh(V))}$$

u(t) périodique ==> $V_p(t)$ **est une unique solution périodique G.A.S**

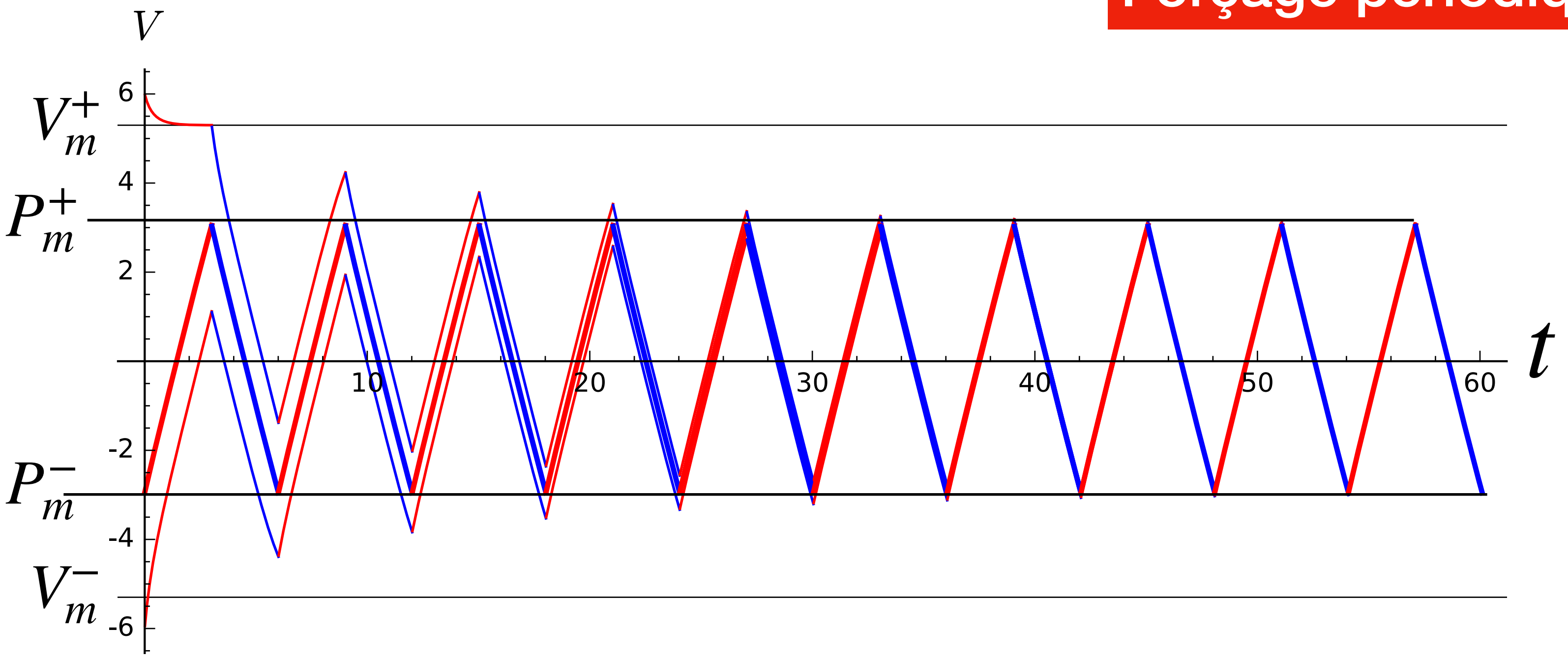
$P(V_0) = V(T, V_0)$ **est croissante, telle que $P' < 1$**

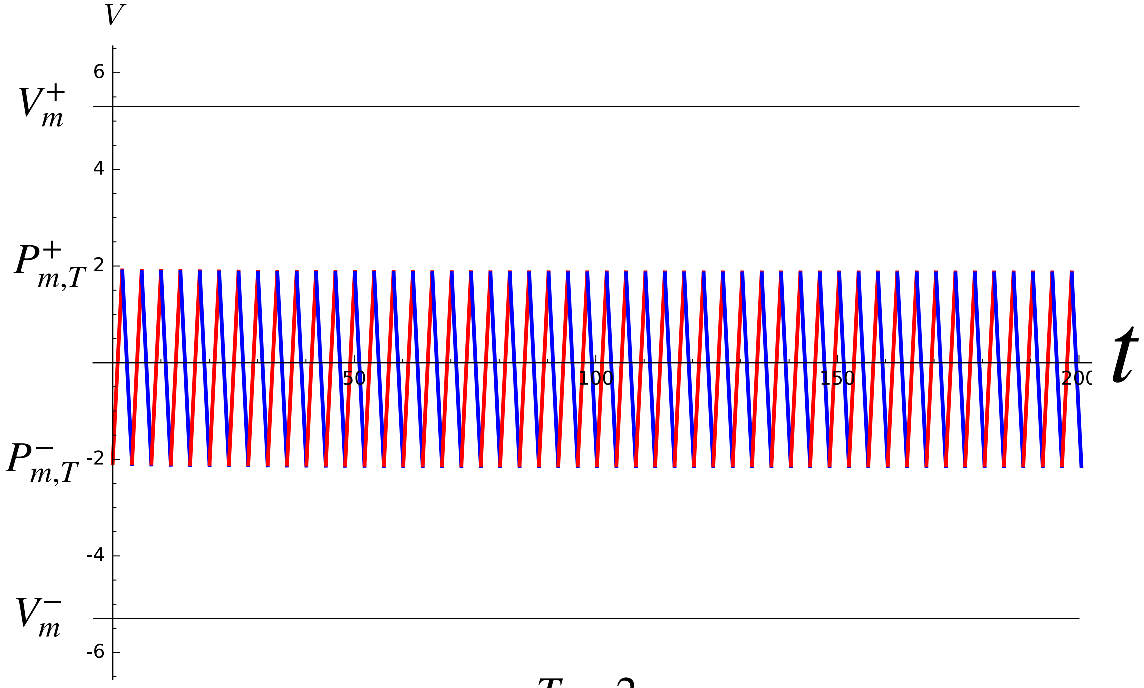




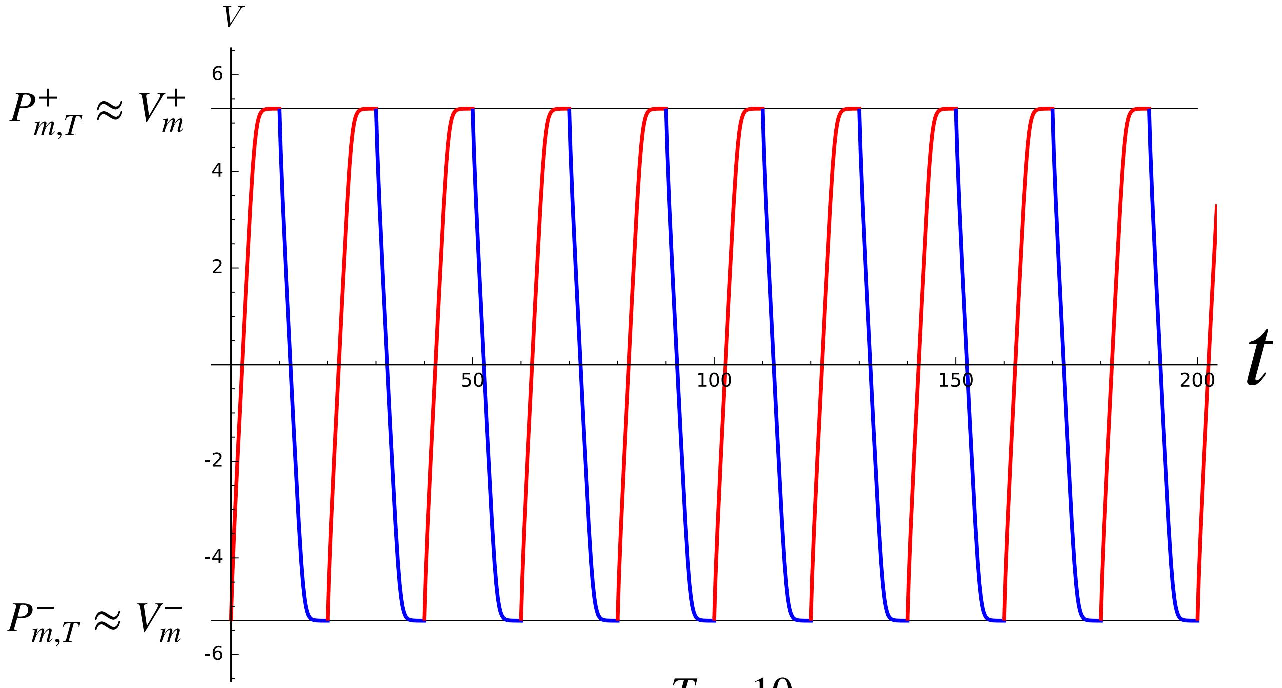
$$\overline{\Delta}(\varepsilon, mT) = \int_0^T 2(m\cosh(V_p(t)) - m - \varepsilon)dt$$

Forçage périodique constant par morceaux



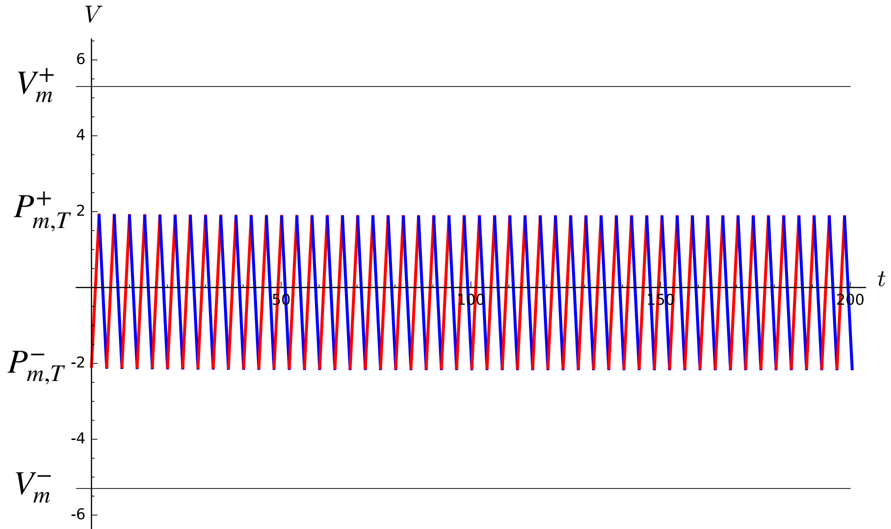


$T = 2$

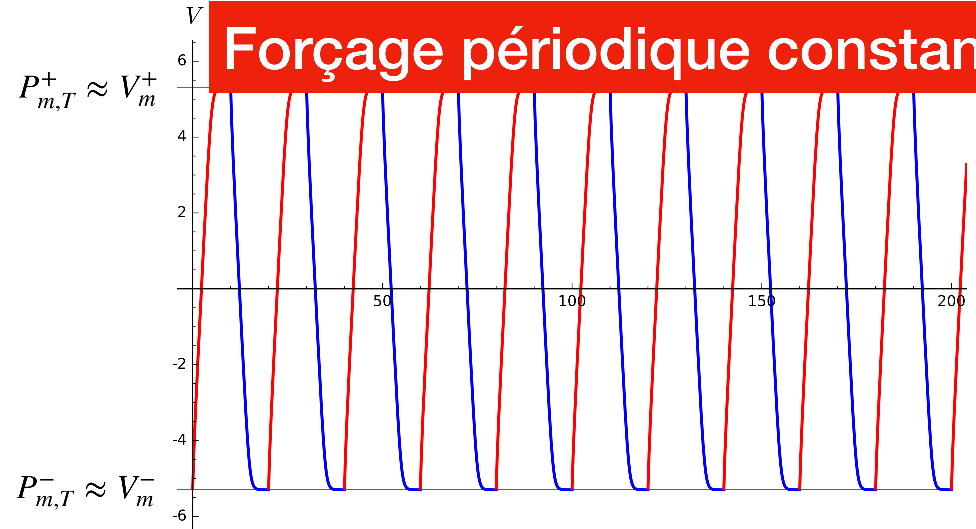


$T = 10$

Forçage périodique constant par morceaux



$T = 2$

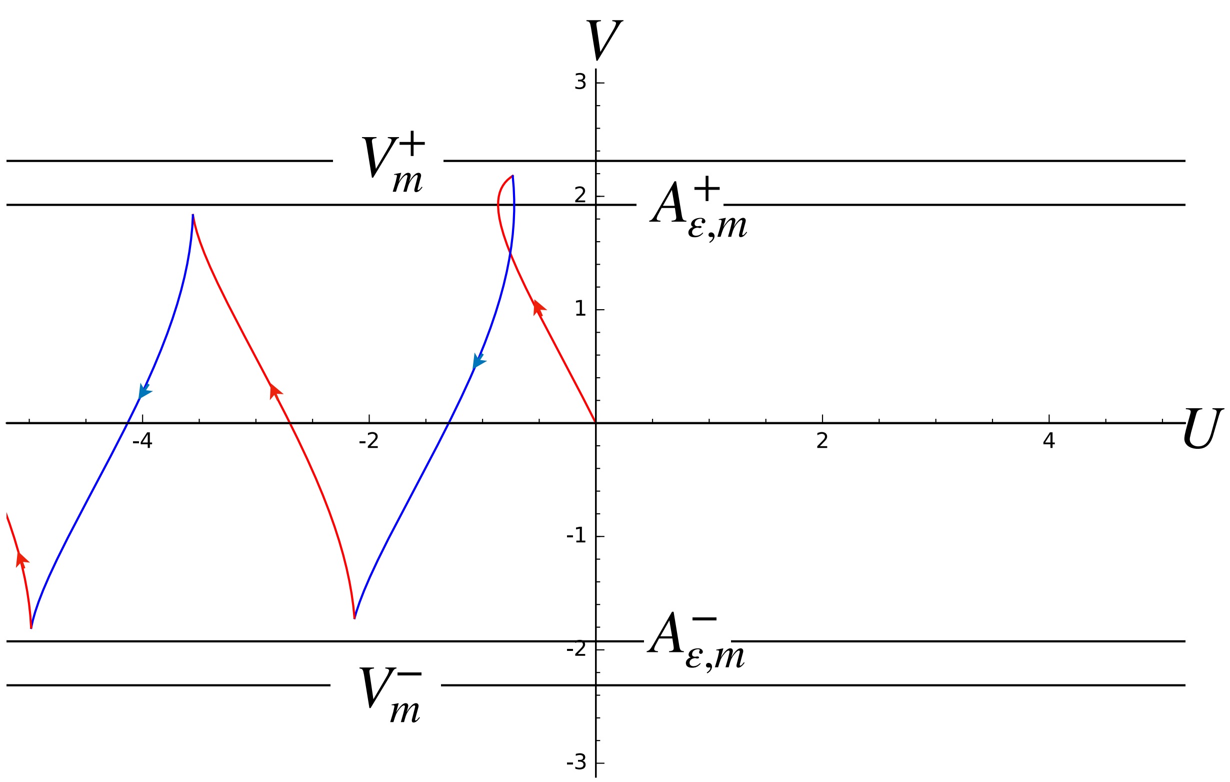


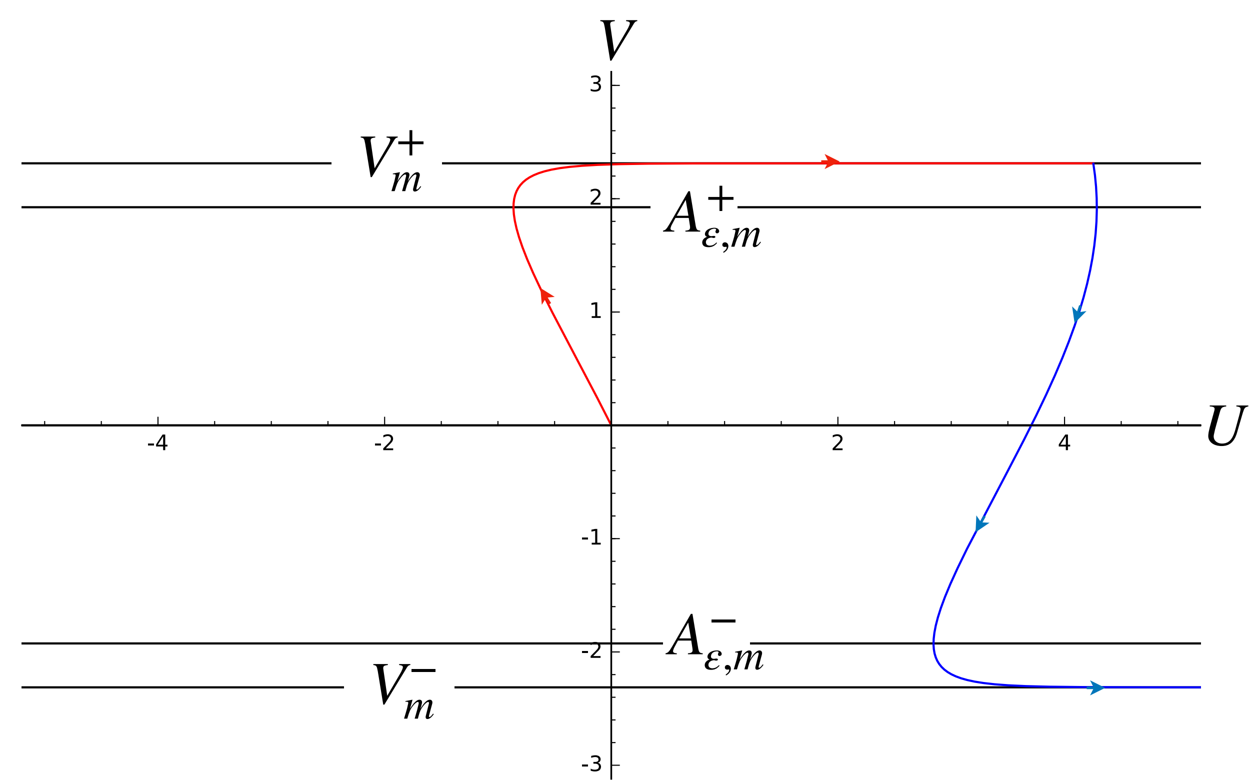
$T = 10$

$$\boxed{U(t) = U_0 + \int_0^t 2\,(m\cosh(V(s) - m - \varepsilon)\,ds}$$

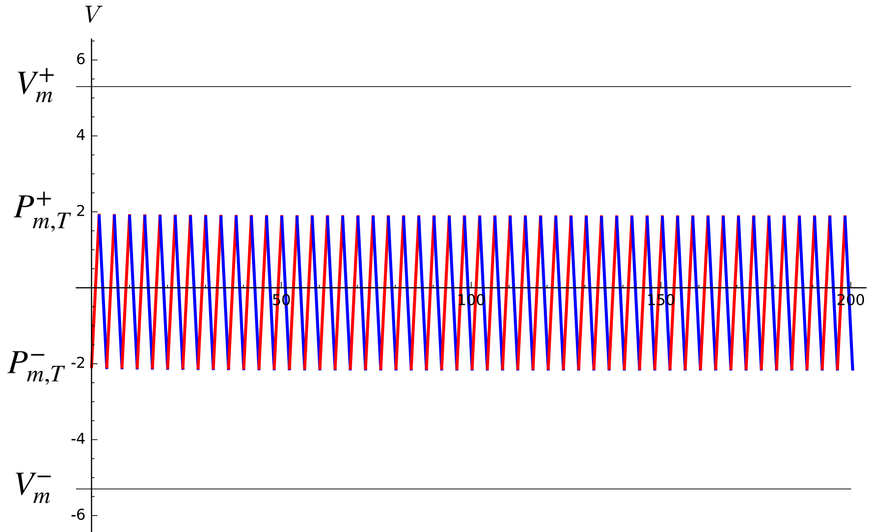
$$A_{\varepsilon,m}^{\pm} = \operatorname{arccosh}\left(1 - \frac{\varepsilon}{m}\right)$$

$$m < \frac{1-\varepsilon^2}{2\varepsilon} \Longrightarrow [A^-, A^+] \subset [V^-, V^+]$$

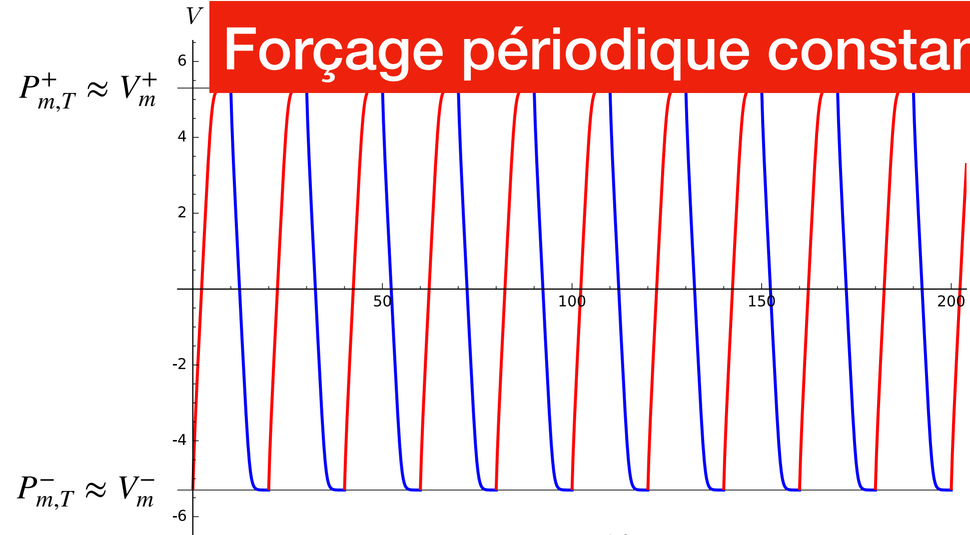




Forçage périodique constant par morceaux



$T = 2$

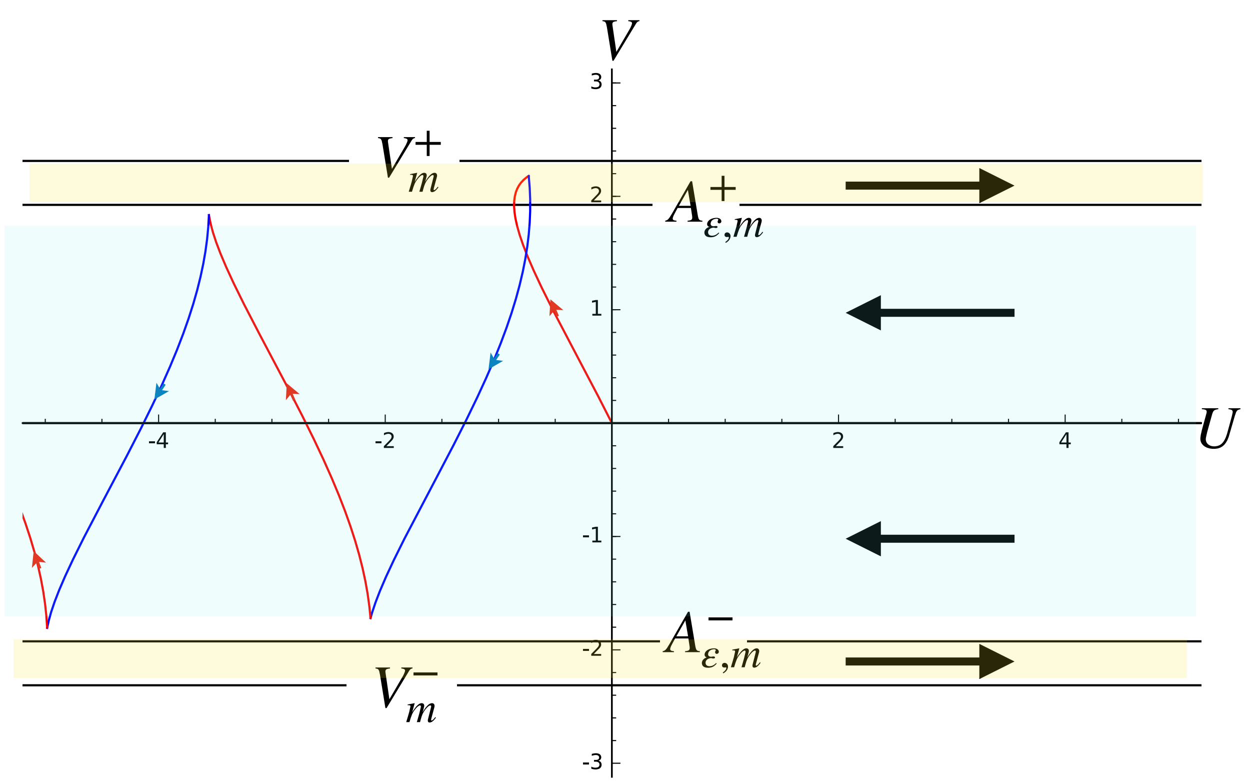


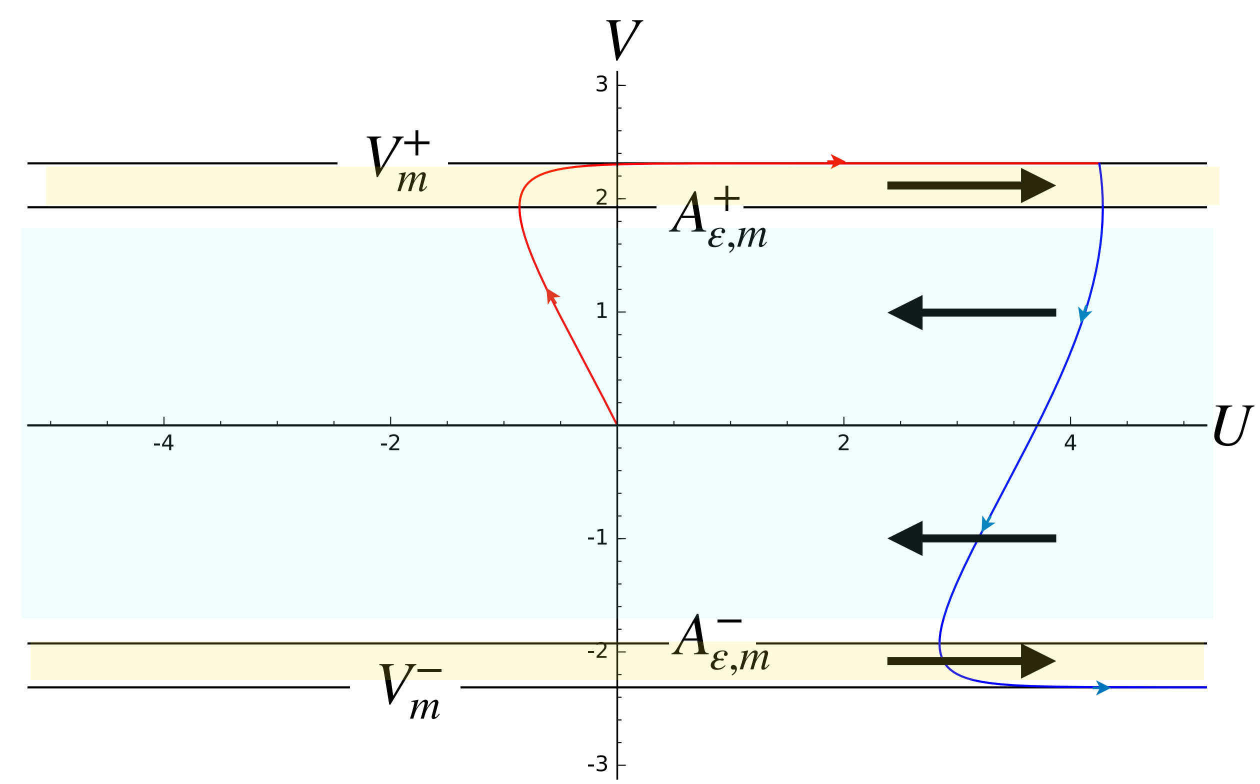
$T = 10$

$$\boxed{U(t) \;=\; U_0 + \int_0^t 2\,(m\cosh(V(s) - m - \varepsilon)\,ds}$$

$$A^{\pm}_{\varepsilon,m} = \operatorname{arccosh}\left(1 - \frac{\varepsilon}{m}\right)$$

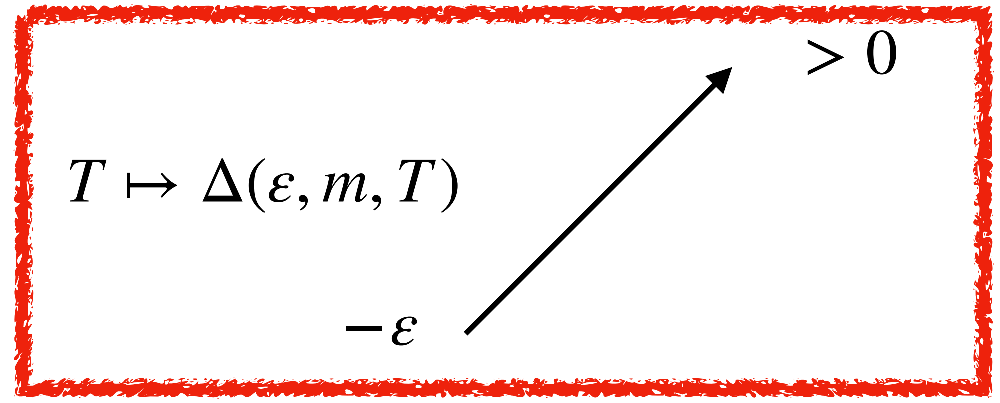
$$m < \frac{1-\varepsilon^2}{2\varepsilon} \Longrightarrow [A^-, A^+] \subset [V^-, V^+]$$





$$2\Delta(\varepsilon, m, T) = \lim_{t\to\infty} \frac{U(t)}{t} = \int_0^{2T} 2(m\cosh(V_p(t)) - m - \varepsilon)dt$$

$$m < \frac{1-\varepsilon^2}{2\varepsilon} \Longrightarrow [A^-, A^+] \subset [V^-, V^+] \qquad \Longrightarrow$$



$$\Downarrow$$

$$\Delta(\varepsilon, m, T^*(m)) = 0$$

C’est un résultat que Katriel démontre de façon générale pour n sites

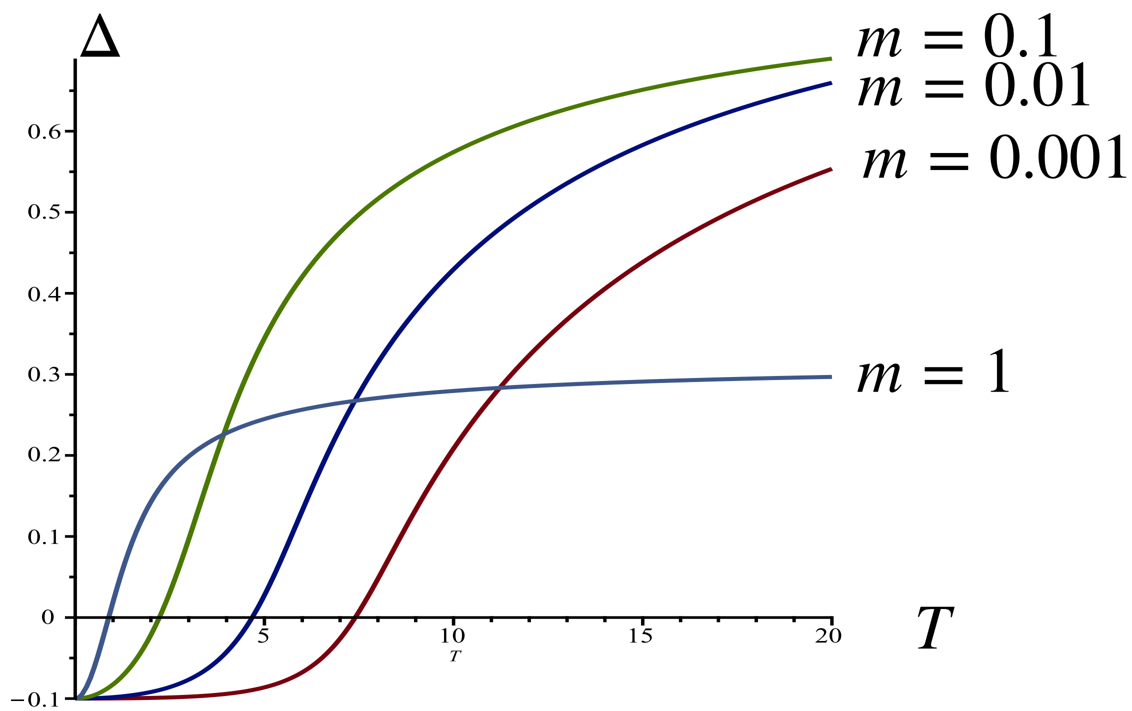
Forçage périodique constant par morceaux

$$\Delta(\varepsilon, m, T) = \lim_{t\to\infty} \frac{U(t)}{t} = \int_0^{2T} 2(m\cosh(V_p(t)) - m - \varepsilon)dt$$

$$m < \frac{1-\varepsilon^2}{2\varepsilon} \Longrightarrow [A^-, A^+] \subset [V^-, V^+] \qquad \Longrightarrow \qquad \boxed{T \mapsto \Delta(\varepsilon, m, T) \quad -\varepsilon \nearrow \;> 0}$$

$$\Downarrow$$

$$\Delta(\varepsilon, m, T^*(m)) = 0$$



m est de l'ordre de exp(-T)

Forçage aléatoire
P.D.M.P.
Piecewise Deterministic Markov Processes

Forçage aléatoire

Les PDMP ont été introduits par :

[8] Davis, M. H. *Piecewise deterministic Markov processes : a general class of non diffusion stochastic models.* Journal of the Royal Statistical Society : Series B (Methodological), 46(3), 353-376.(1984)

Entre autres applications à la dynamique des populations:

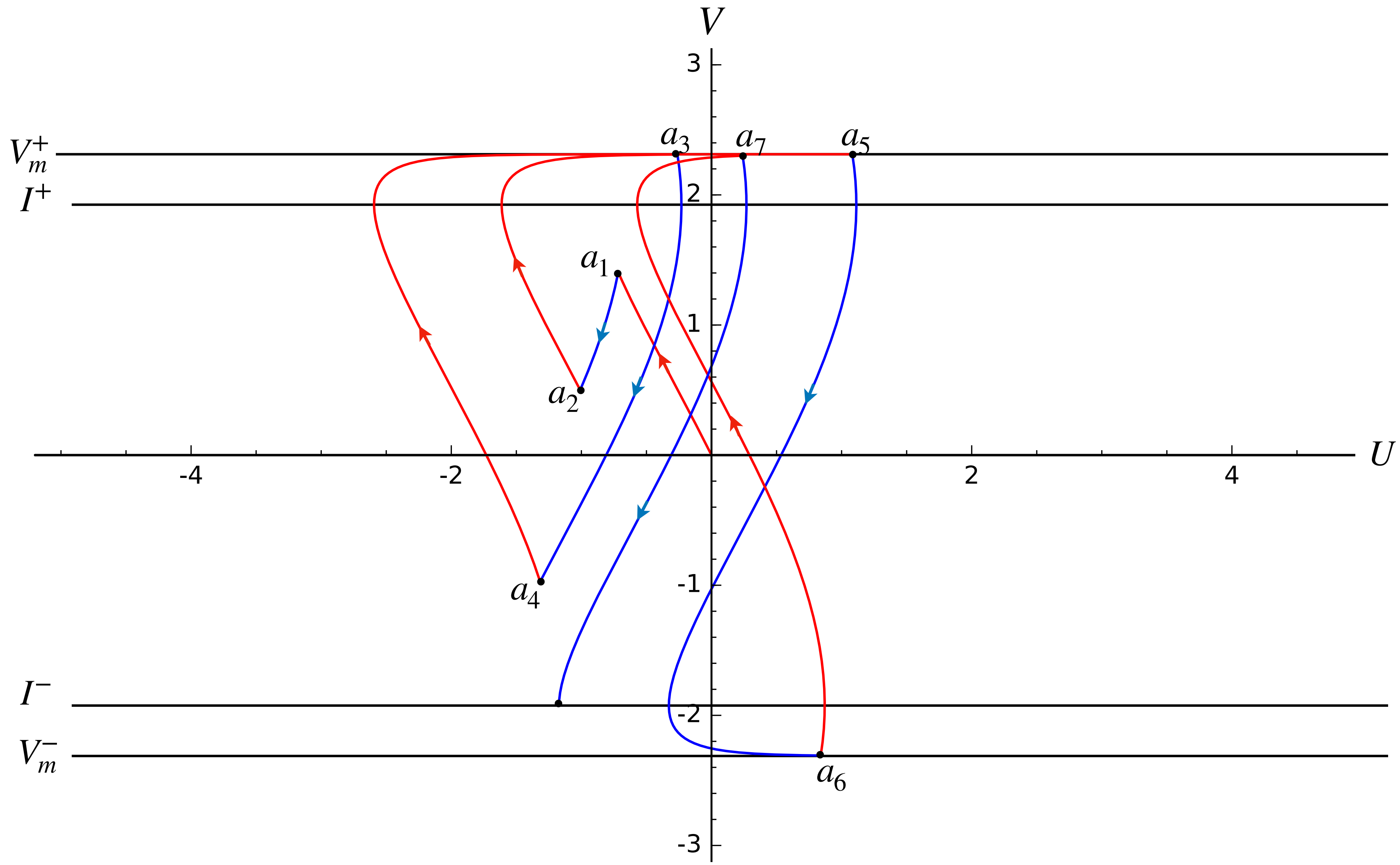
[4] Benaïm, M. and Le Borgne, S. and Malrieu, F. and Zitt, P.- A. *Qualitative properties of certain piecewise deterministic Markov processes* = Ann. Inst. Henri Poincaré Probab. Stat. Annales de l'Institut Henri Poincaré, ProbabilitÈs et Statistiques, (2015)

[5] Benaïm, M., & Lobry, C. *Lotka–Volterra with randomly fluctuating environments or "How switching between beneficial environments can make survival harder".* The Annals of Applied Probability, 26(6), 3754-3785. (2016)

[6] Benaïm, M. and Strickler, E. *Random Switching between Vector Fields Having a Common Zero* Ann. Appl. Probab., 29(1), 326-375 (2019)

[12] Hening, Alexandru and Strickler, Edouard *On a predator-prey system with random switching that never converges to its equilibrium* SIAM Journal on Mathematical Analysis 51(5), 3625–3640 (2019)

[13] Hurth, Tobias and Strickler, Edouard *Randomly Switched ODES with nonexponential switching times.* Preprint, 2021

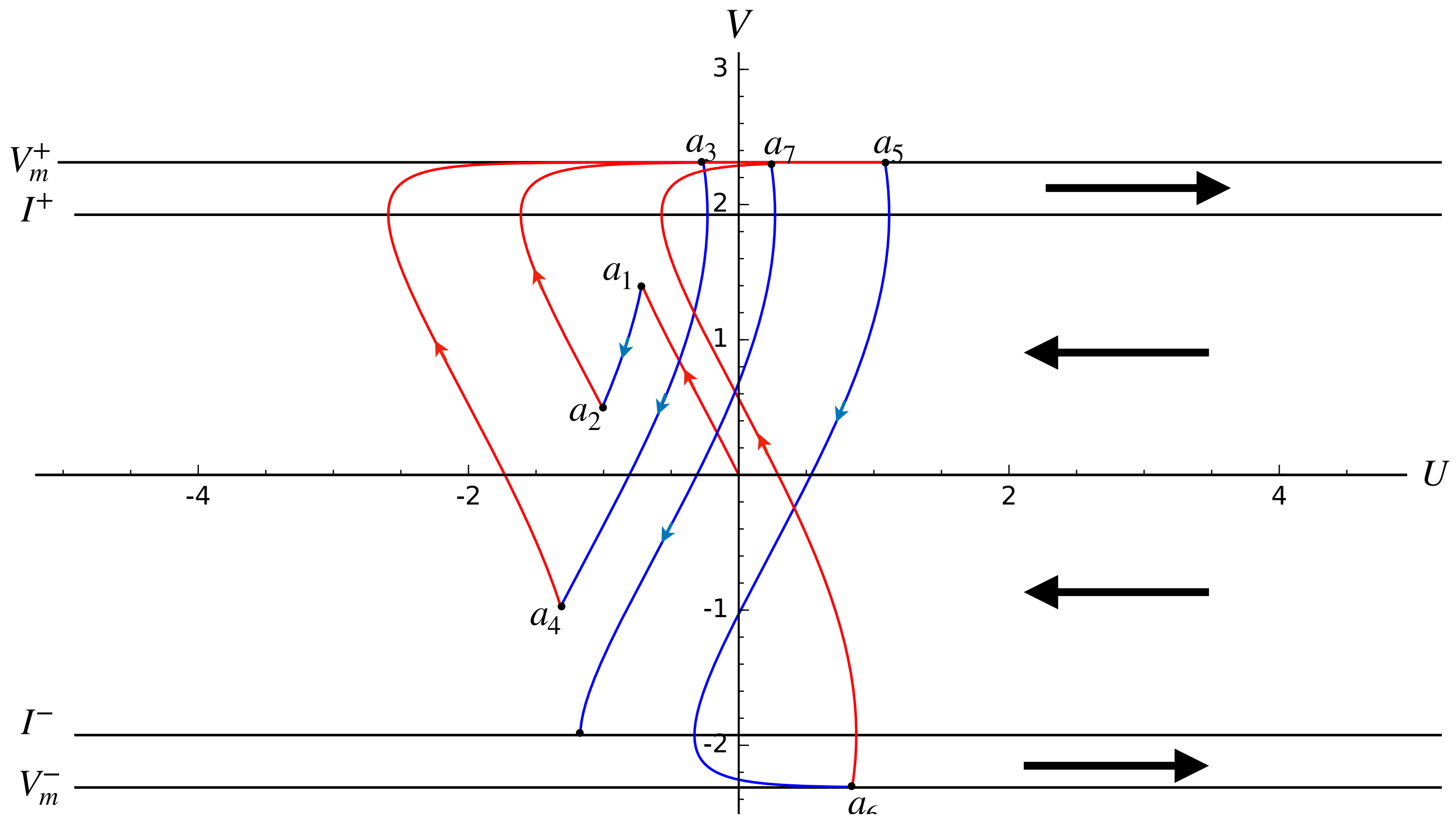


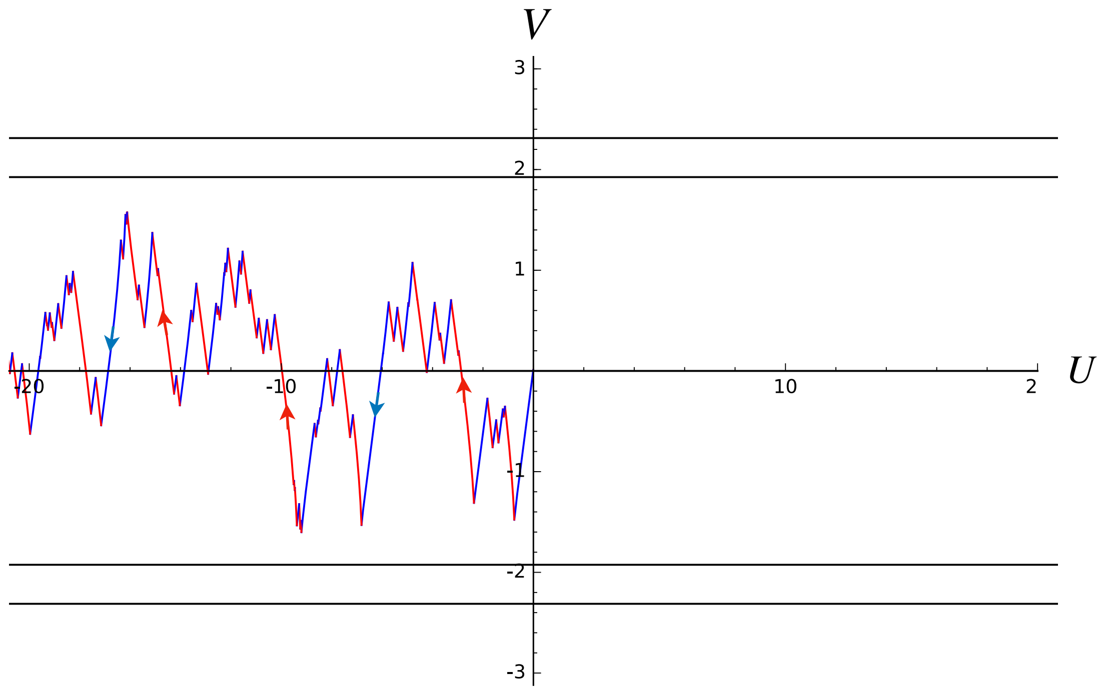
Piecewise Deterministic Markov Processes (PDMP) :

Des variables aléatoires indépendantes $T_1, T_2, \cdots, T_n, \cdots$ suivant une loi exponentielle

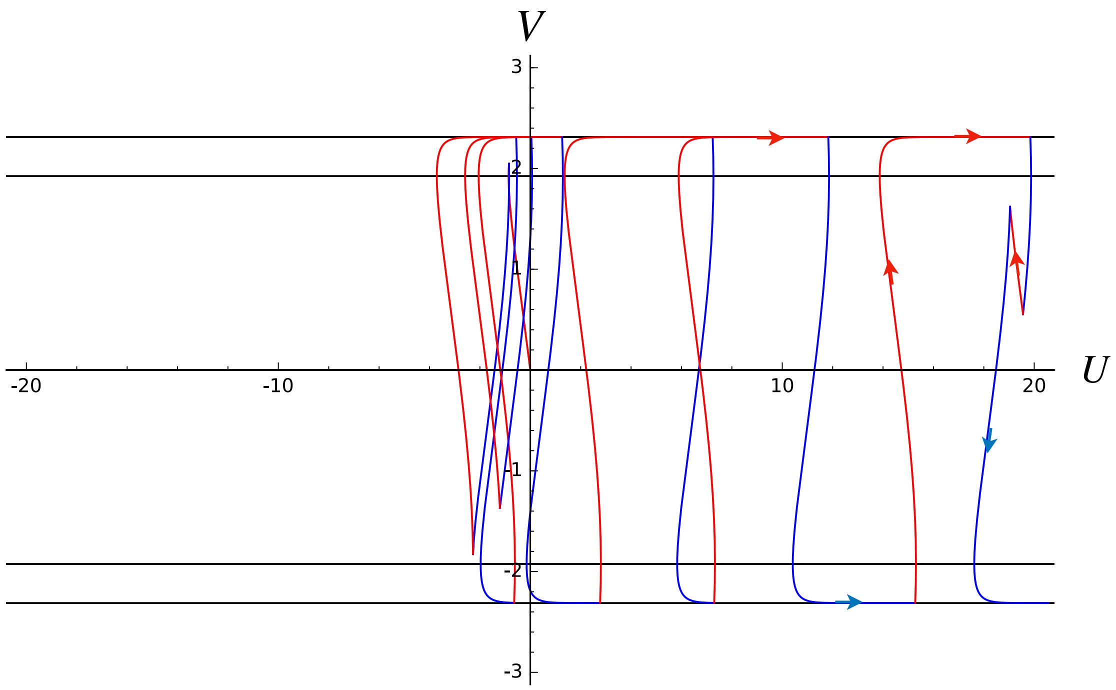
Définissent un processus de Markov U_t, V_t

Forçage aléatoire





$T = 0.2$



$T = 5$

$$T_i \sim \mathscr{E}(\mu) \Longleftrightarrow P(T_i \leq x) = \frac{1}{\mu}\int_0^x e^{-\mu s} ds \qquad E(T_i) = \frac{1}{\mu}$$

$$T_i \sim \mathscr{E}(\frac{1}{T}) \Longleftrightarrow P(T_i \leq x) = T\int_0^x e^{-\frac{1}{T}s} ds \qquad E(T_i) = T$$

„Périodique en moyenne T"

Forçage aléatoire

Le processus V_t a une unique probabilité stationnaire μ définie sur $[V_m^-, V_m^+]$ qui est qui est calculable explicitement telle que pour toute fonction, p. s. Φ

$$\lim_{t\to+\infty} \frac{\int_0^t \Phi(V_t)}{t} = \int_{[V_m^-, V_m^+]} \phi(s)\, d\mu(s) \qquad a.s.$$

$$\overline{\Delta}(\varepsilon, m, T) = \lim_{t\to\infty} \frac{U_t}{t} = \frac{1}{t}\int_0^t 2(m\cosh(V_t) - m - \varepsilon)dt = \int_{[V_m^-, V_m^+]} 2(m\cosh(s) - m - \varepsilon)d\mu(s)$$

$$\overline{\Delta}(\varepsilon, m, T) = \frac{1}{t}\int_0^t 2(m\cosh(V_t) - m - \varepsilon)dt = \int_{[V_m^-, V_m^+]} 2(m\cosh(s) - m - \varepsilon)d\mu(s)$$

$$\boldsymbol{\rho}^h_{m,T}(v) = \frac{C(m)}{|F^h_m(v)|}\left(\frac{e^{V_m^+} - e^v}{e^v + e^{V_m^-}}\,\frac{e^v - e^{V_m^-}}{e^v + e^{V_m^+}}\right)^{\frac{1}{2T\sqrt{m^2+1}}}$$

$$\int_{V_m^-}^{V_m^+} \rho_m^+(v) + \rho_m^-(v)\, dv = 1.$$

formules explicites pour

$$\Longrightarrow \quad \overline{\Delta}(\varepsilon, m, T)$$

$\Downarrow$ étude asymptotique

Forçage aléatoire

Le processus V_t a une unique probabilité stationnaire μ définie sur $[V_m^-, V_m^+]$ qui est qui est calculable explicitement telle que pour toute fonction. p. s. Φ

Cas déterministe

$$\Delta(\varepsilon, m, T) = \lim_{t\to\infty} \frac{U(t)}{t} = \int_0^{2T} 2(m\cosh(V_p(t) - m - \varepsilon)dt$$

$$\overline{\Delta}(\varepsilon, m, T) = \lim_{t\to}$$

$$\overline{\Delta}(\varepsilon, m, T) = \frac{1}{t}\int_0^t 2(m\cosh(V_t) - m - \varepsilon)dt = \int_{[V_m^-, V_m^+]} 2(m\cosh(s) - m - \varepsilon)d\mu(s)$$

$$\boldsymbol{\rho}_{m,T}^{h}(v) = \frac{C(m)}{|F_m^h(v)|}\left(\frac{e^{V_m^+} - e^v}{e^v + e^{V_m^-}}\frac{e^v - e^{V_m^-}}{e^v + e^{V_m^+}}\right)^{\frac{1}{2T\sqrt{m^2+1}}}$$

$$\int_{V_m^-}^{V_m^+} \rho_m^+(v) + \rho_m^-(v)\, dv = 1.$$

formules explicites pour

$\Longrightarrow$ $\overline{\Delta}(\varepsilon, m, T)$

$\Downarrow$ étude asymptotique

Forçage aléatoire

Theorem

1. Pour tout $m > 0$,

$$\lim_{T\to 0} \overline{\Delta}(\varepsilon, m, T) = -2\varepsilon < 0.$$

2. Pour tout $m > 0$,

$$\lim_{T\to\infty} \overline{\Delta}(\varepsilon, m, T) = 2\left(\sqrt{m^2+1} - m - \varepsilon\right).$$

en particulier

Existence de l'inflation pour T grand

$$\lim_{T\to\infty} \overline{\Delta}(\varepsilon, m, T) > 0 \quad \Leftrightarrow \quad m < \frac{1-\varepsilon^2}{2\varepsilon}.$$

3. Pour tout $T > 0$,

mais pour m ni trop grand ni trop petit

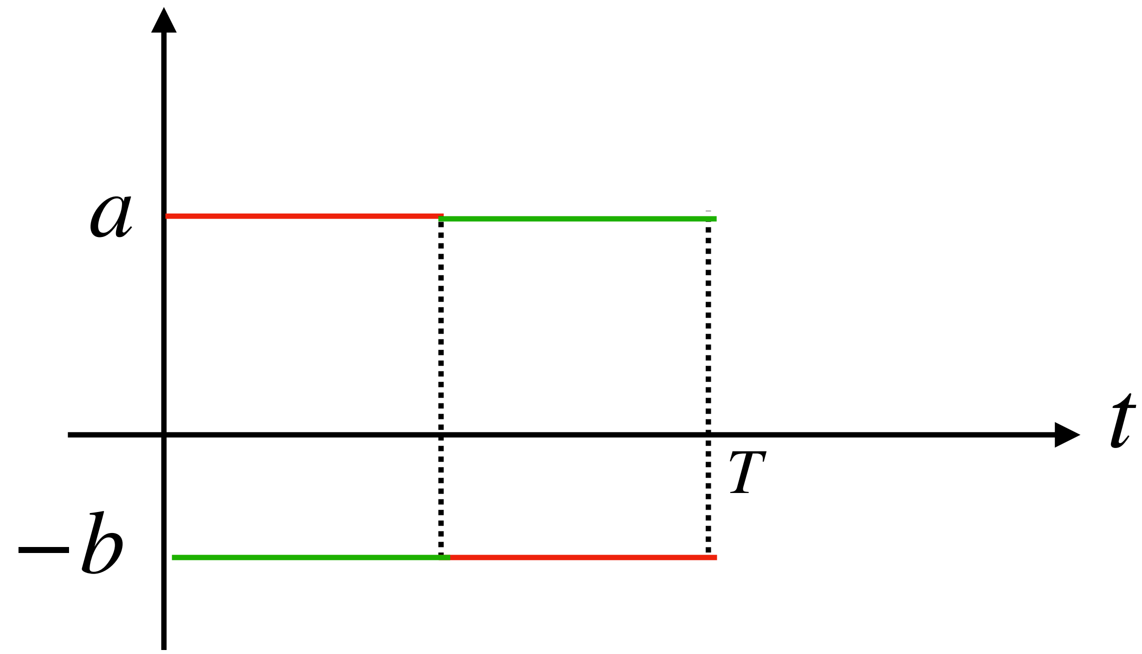
$$\lim_{m\to 0} \overline{\Delta}(\varepsilon, m, T) = \lim_{m\to\infty} \overline{\Delta}(\varepsilon, m, T) = -2\varepsilon.$$

On retrouve la même conclusion que dans le cas déterministe

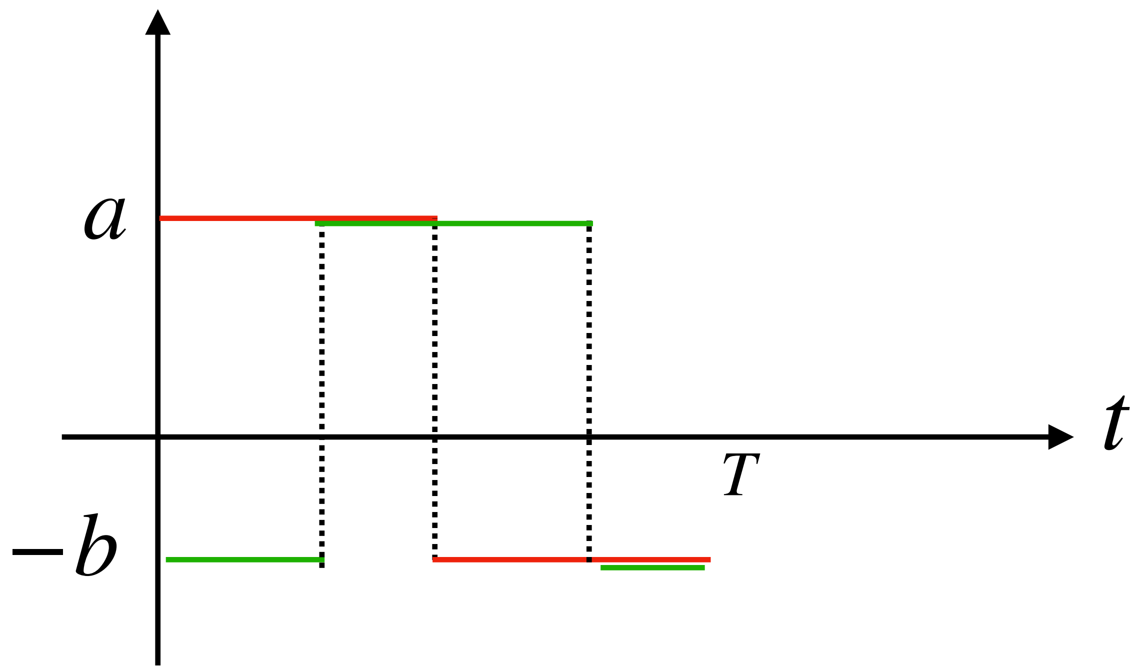
Extensions

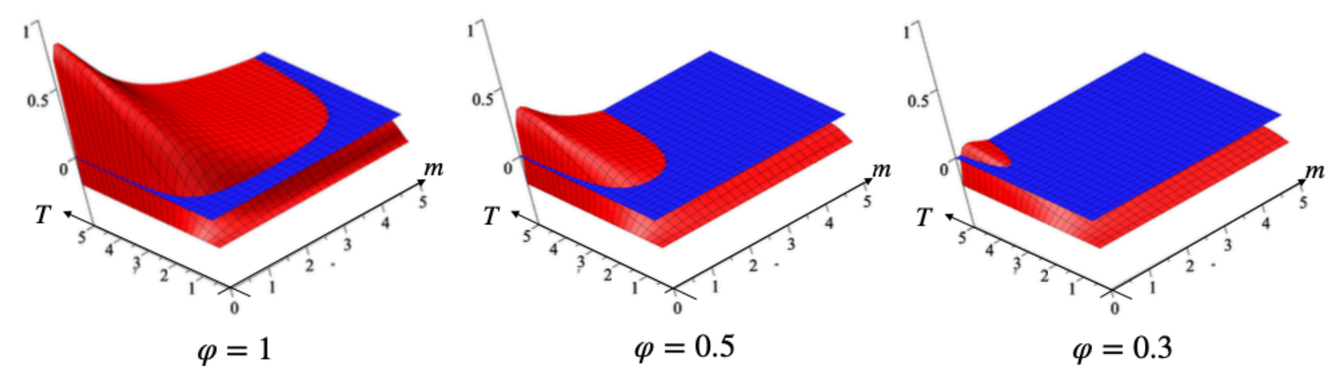
Le modèle (+a-b)

Opposition de phase



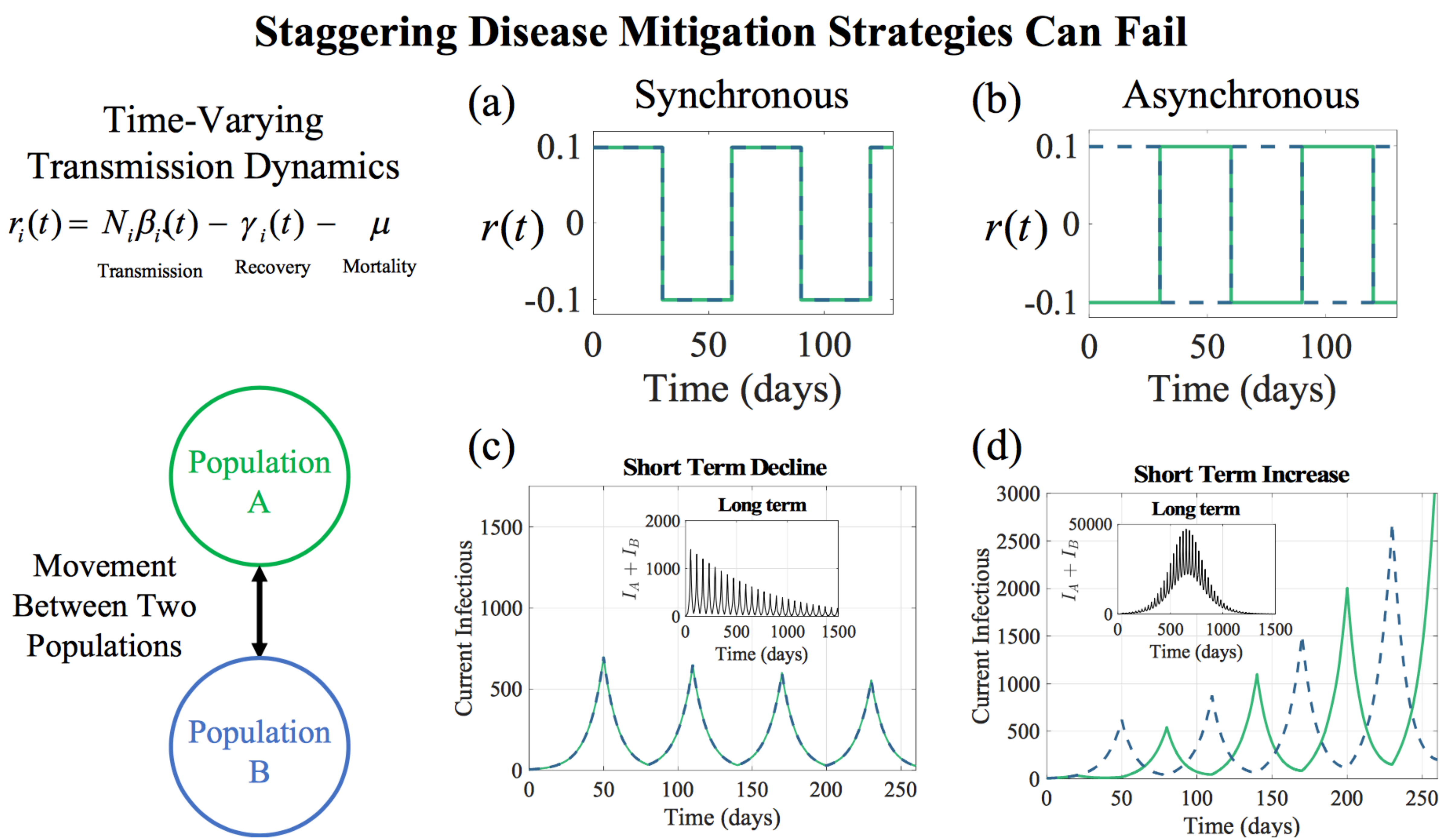
Déphasage partiel

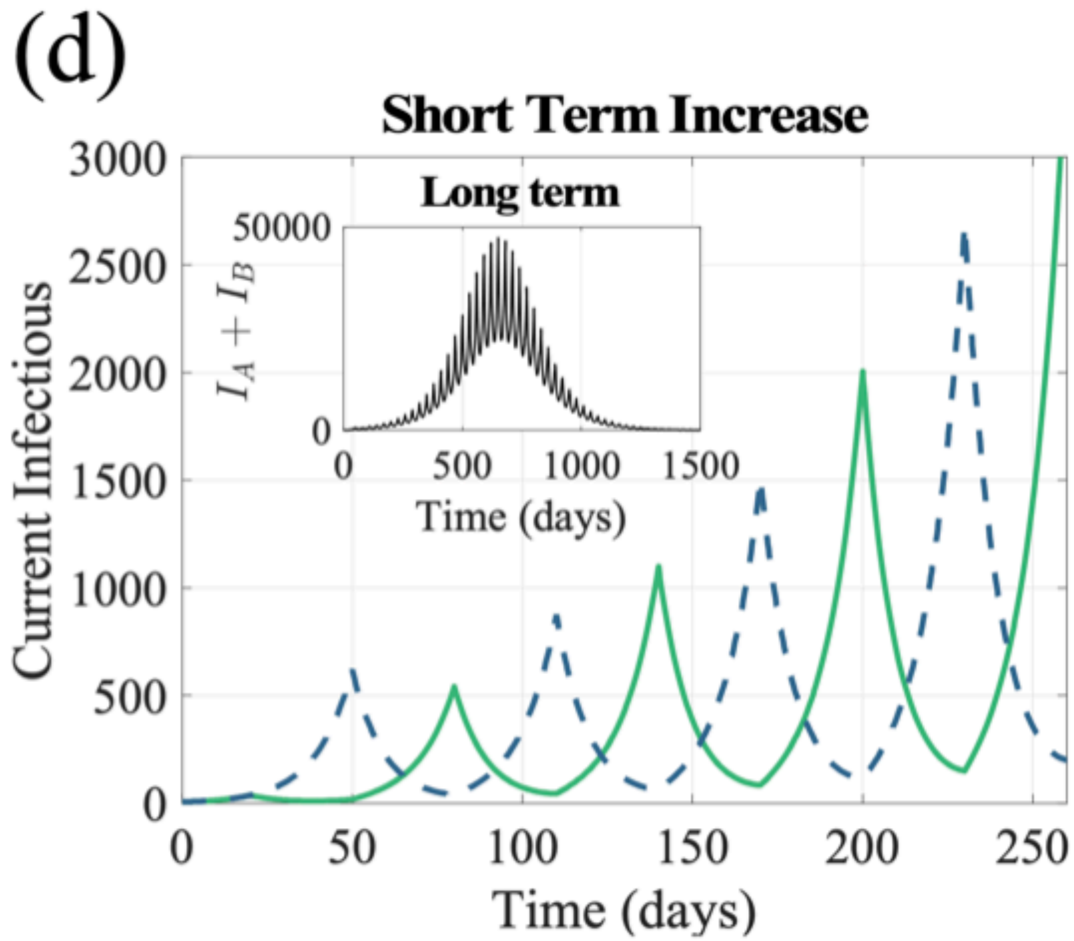


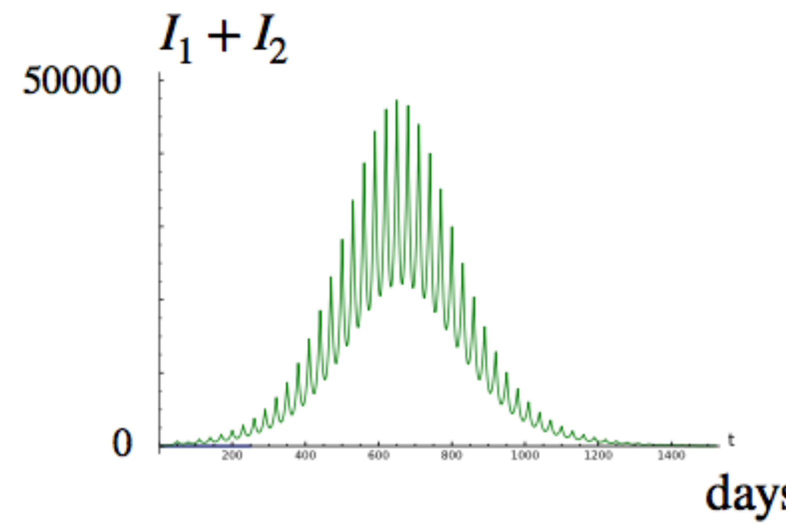


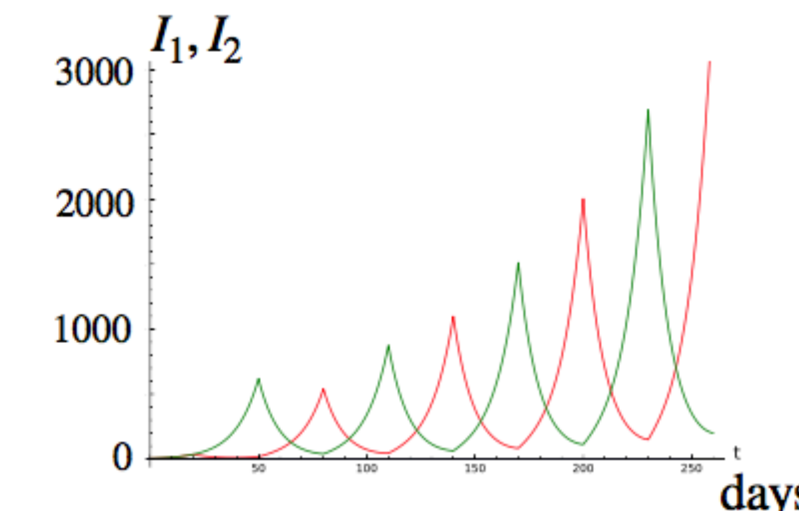
Retour sur l'introduction

Figures







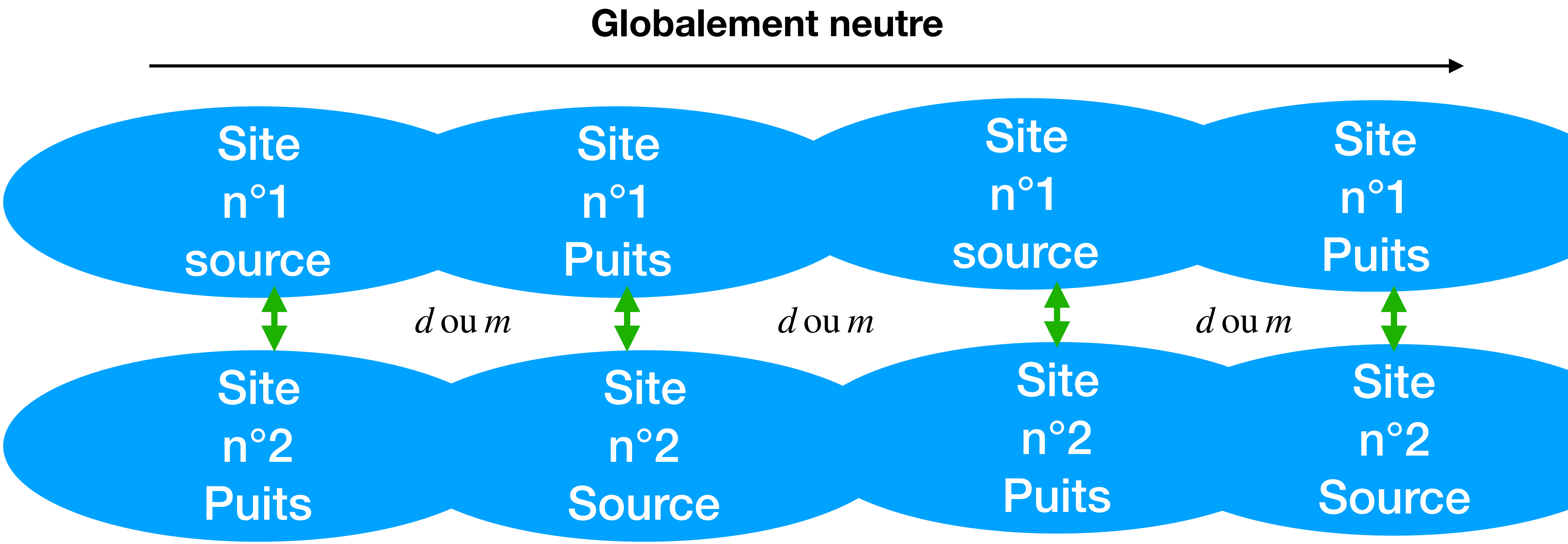


Même modèle simulé par nous

$\beta_n N = 0.1988$	$\gamma_n = 0.098$	$\mu_n = 0.002$
$\beta_s N = 0.0288$	$\gamma_s = 0.128$	$\mu_s = 0.002$

$\beta_n N - (\gamma_n + \mu_n) = 0.0988$
$\beta_s N - (\gamma_s + \mu_s) = -0.1012$

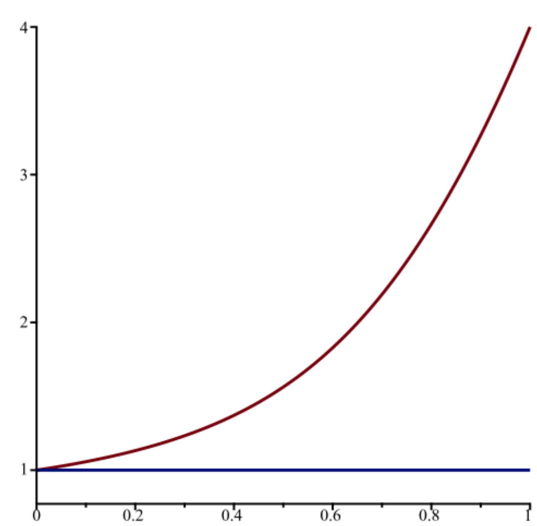
Le secret

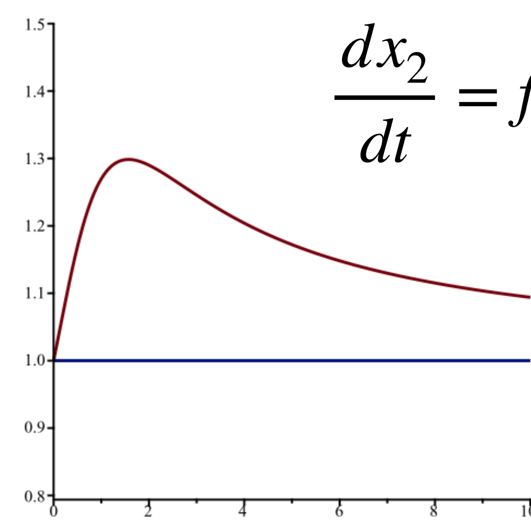


$$x_1(n+1) = [f(x_1(n), x_2(n))](1-d)$$

$$x_2(n+1) = [f(x_1(n), x_2(n))]d$$

$$d \in [0,1]$$



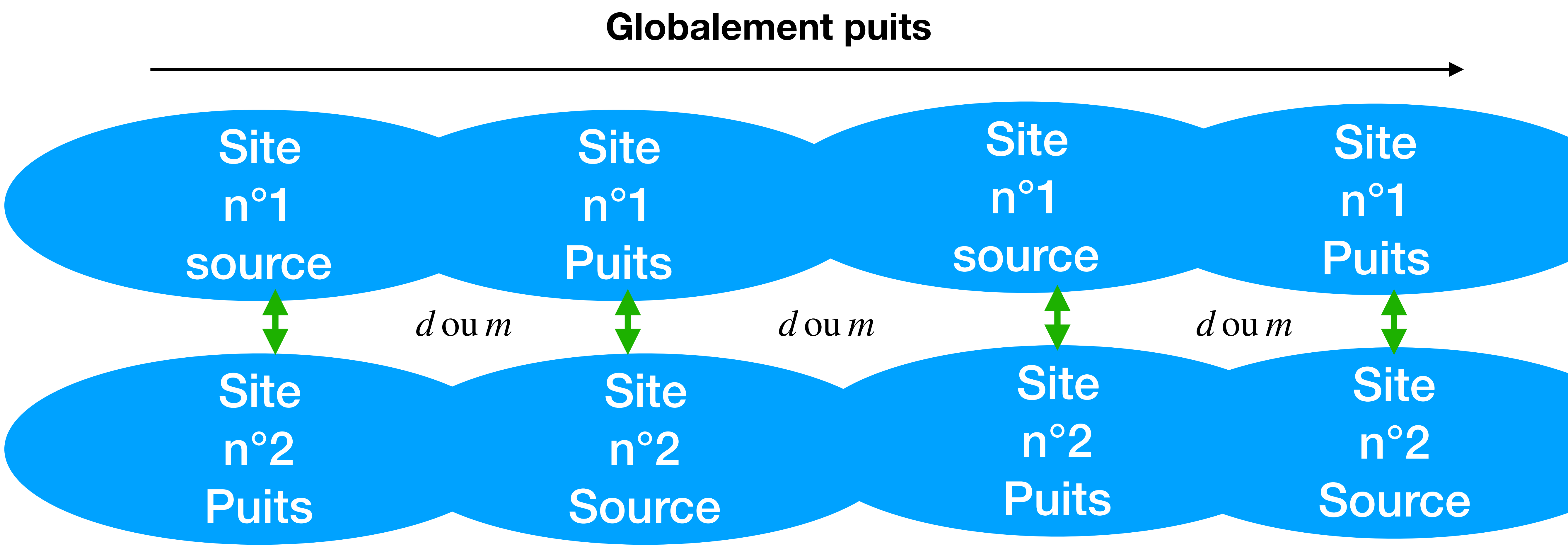


$$\frac{dx_1}{dt} = f(x_1, x_2) + m(x_2 - x_1)$$

$$\frac{dx_2}{dt} = f(x_1, x_2) + m(x_1 - x_2)$$

$$m \in [0, +\infty)$$

FIGURE 4 – Modèle "alternatif" : Discret (à gauche), continu (à droite)



$$x_1(n+1) = [f(x_1(n), x_2(n))](1-d)$$

$$x_2(n+1) = [f(x_1(n), x_2(n))]d$$

$$d \in [0,1]$$

$$\frac{dx_1}{dt} = f(x_1, x_2) + m(x_2 - x_1)$$

$$\frac{dx_2}{dt} = f(x_1, x_2) + m(x_1 - x_2)$$

$$m \in [0, +\infty)$$

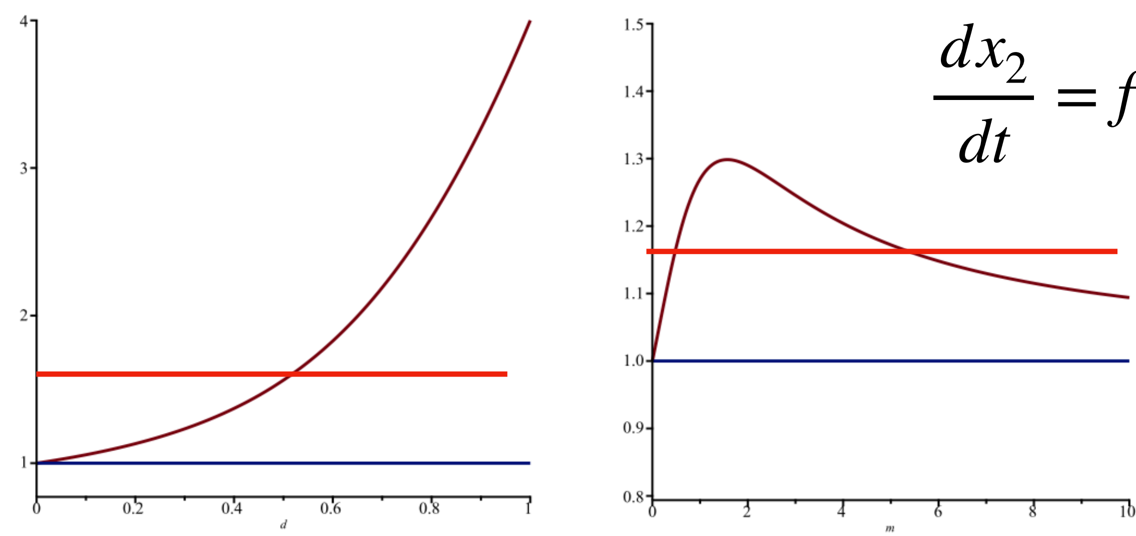
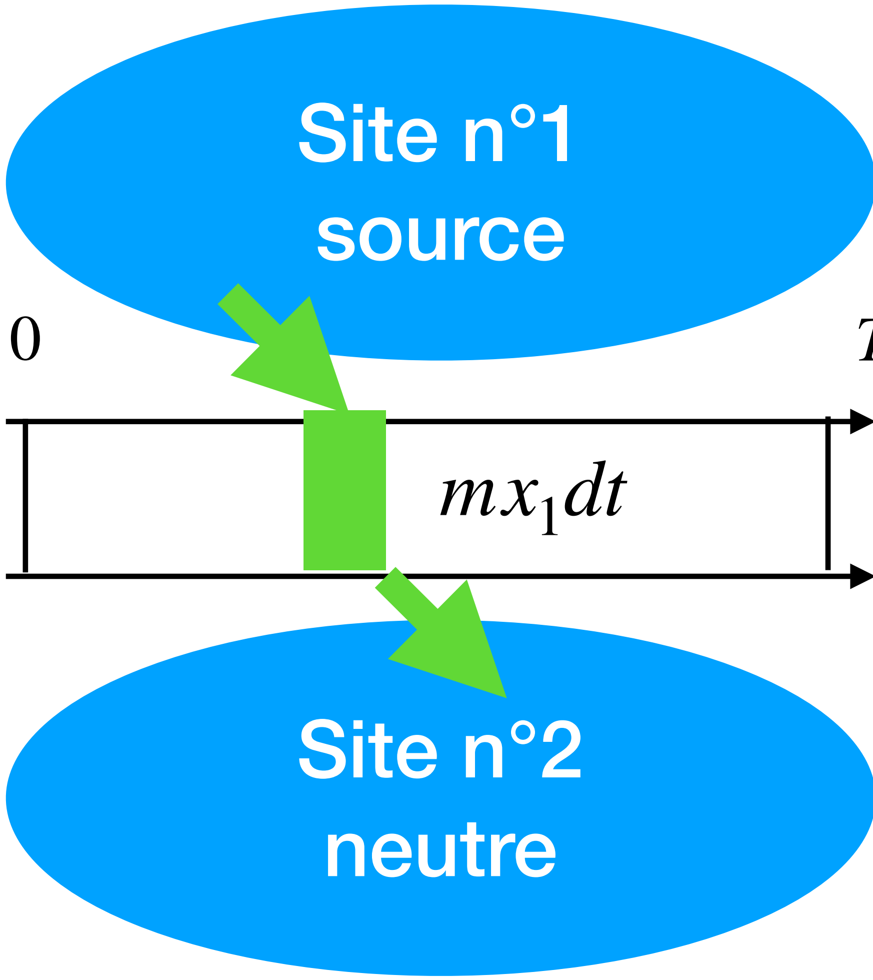
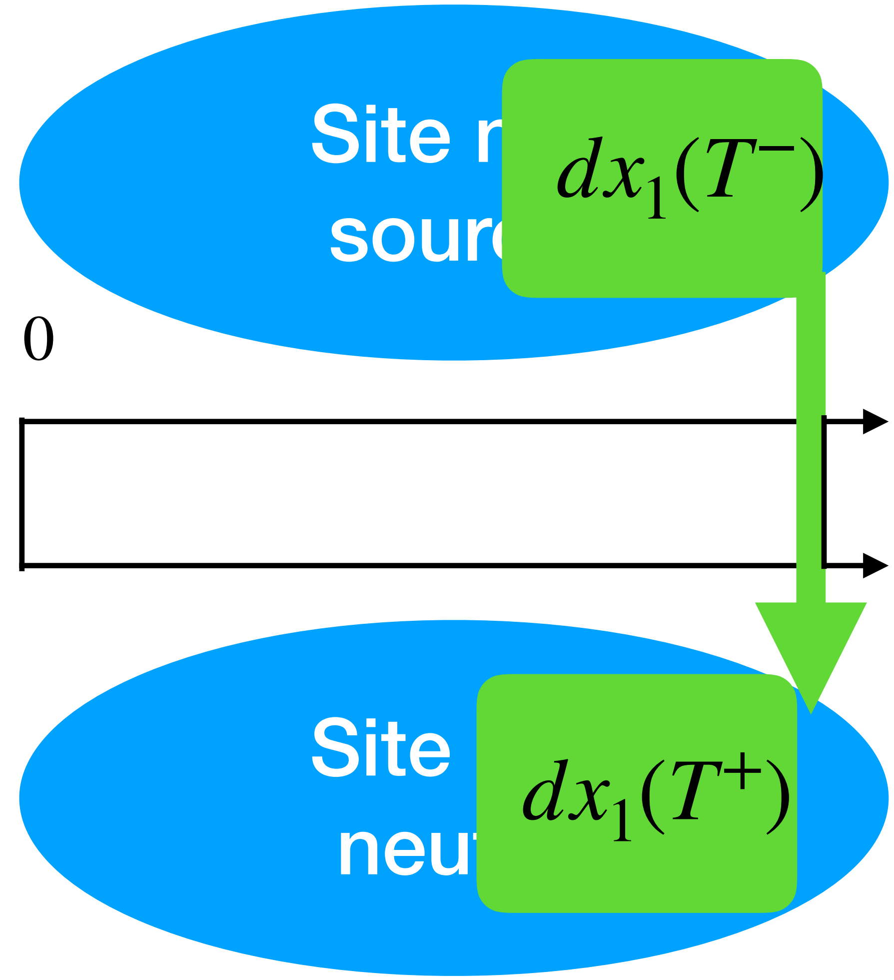


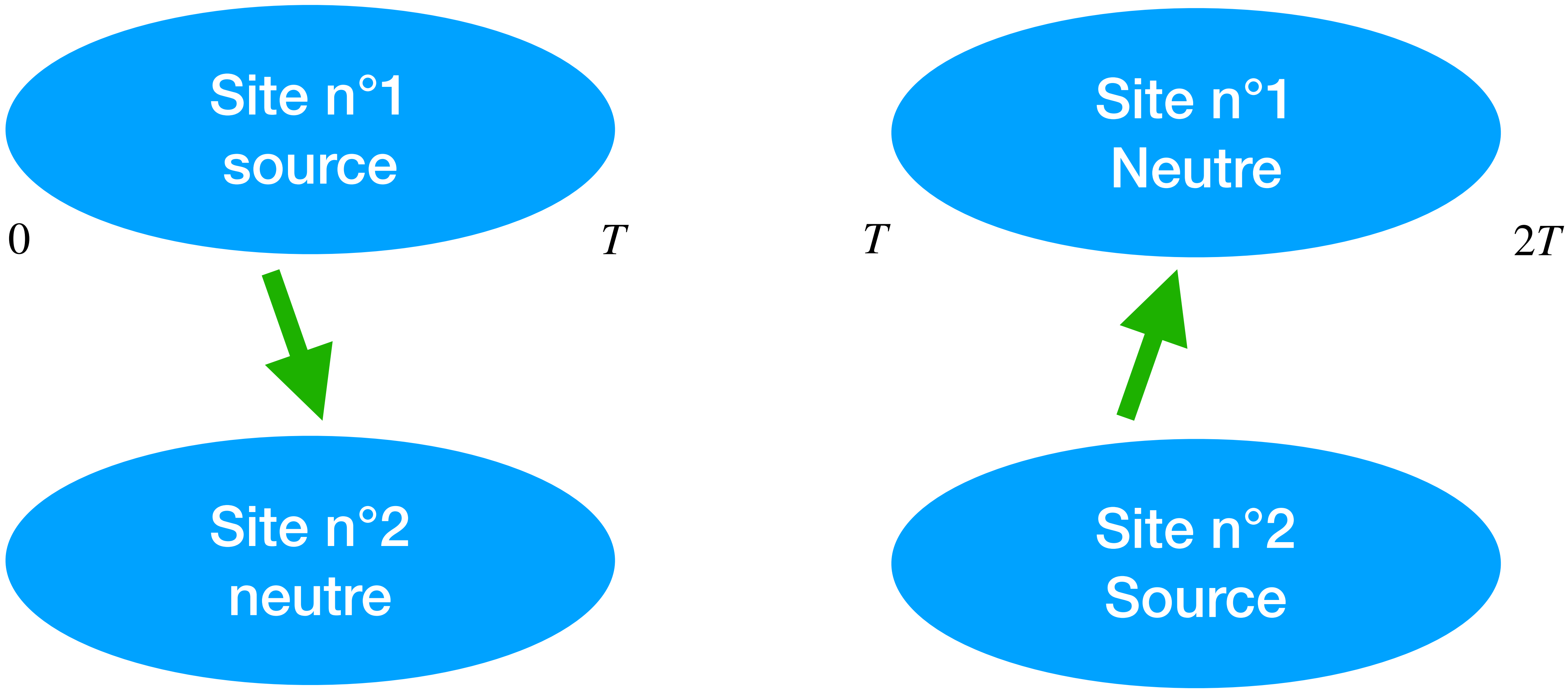
FIGURE 4 – Modèle "alternatif" : Discret (à gauche), continu (à droite)

$$\frac{dx_1}{dt} = (r - m)x_1, \quad r > 0$$



$$\frac{dx_1}{dt} = rx_1, \quad r > 0 \quad x_1(T) = x_1(0)\mathrm{e}^{rT}$$





$$x_1(t) = x_1(0)e^{(r-m)t}$$

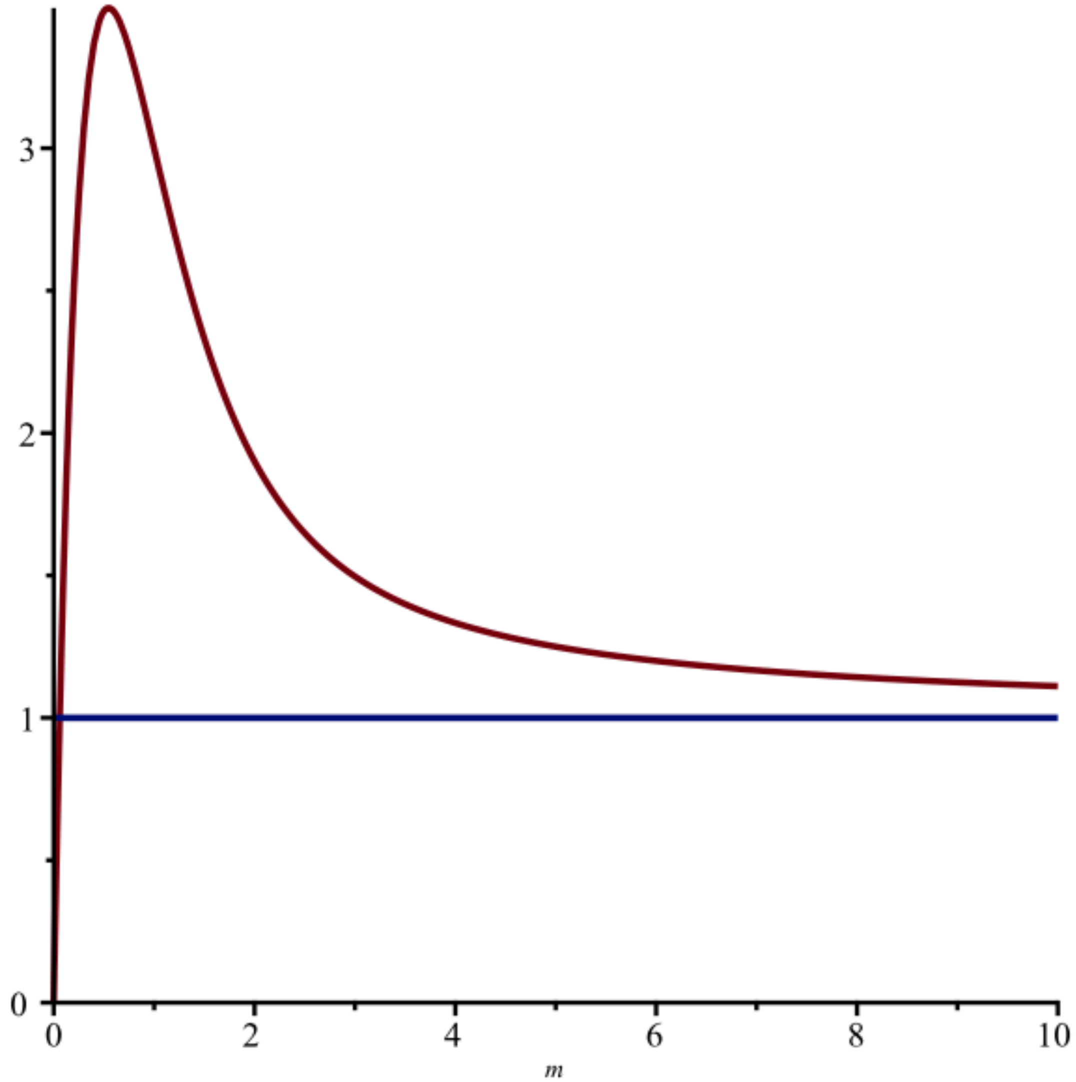
$$x_2(T) = \int_0^T m x_1(0)e^{(r-m)t}dt$$

$$x_2(T) = \frac{m}{r-m}\left(e^{(r-m)T} - 1\right)$$

$$x_1(T) = x_1(0)e^{rt}(1-d)$$

$$x_2(T) = x_1(0)e^{rt}d$$

La disperstion ne change pas
La population totale



$$x_2(T) = \frac{m}{r-m}\left(e^{(r-m)T} - 1\right)$$

Forçage périodique constant par morceaux