Optimal control applied to mathematical model of COVID 19

Amira BOUHALI[*1,2], **Walid BEN ARIBI**[2], **Slimane BEN MILED**[2], **Amira KEBIR**[2,3]

[1]ENIT, Université de Tunis El Manar, 1068, Tunis, Tunisia
[2]BIMS, Institut Pasteur de Tunis, Tunis Belvedère, 1002, Tunis, Tunisia
[3]IPEIT, Université de Tunis, Monfleury, 1008, Tunis, Tunisia

*E-mail : amira.bouhali.2893@gmail.com

Abstract

In this study, we present an epidemic controlled SIRD model of a totally vaccinated population with two types of control strategies: mask wear and screening. The aim of this study is to minimize the number of the Deceased keeping a minimal cost of mask advertising and screening. The model is proved to be well posed and to have an invariant region . Also, a study of the equilibria stability is effected using the basic reproduction number. As for the optimal control analysis, we study the existence of an optimal solution in two different cases: constant and variable mask wear. The characterization of the optimal control is carried out using Pontryagin's minimum principle in both cases. Numerical simulations are conducted for the constant mask wear case with different values of maximal screening for comparison. The findings of the optimal control analysis and numerical simulations both reveal that combining vaccination with the optimal pair of strategies contribute enormously in lowering the number of infected and dead individuals. Although zero infection is not achieved in the population, this study implies that carrying an optimal approach constitutes a major step in controlling the spread of the disease to the barest minimum.

Keywords

Optimal control, Structured models, COVID 19, Basic reproduction number

I INTRODUCTION

Severe acute respiratory syndrome coronavirus 2 commonly known as SARS-CoV-2 is a novel coronavirus that has caused the global pandemic of COVID-19 first reported in Wuhan China. The virus has proved to be very difficult to contain out of the quarantine measures due to its high contagion and lethalness. On the other hand, the economic pressure on the governments has shown how inconvenient the lockdown strategy is and how much required it is to carry on with a normal way of life. The problem has been treated biologically in the first place trying to develop vaccine and treatment. However, despite the implementation of several vaccines and their use, many countries kept registering high numbers of deaths and infections. For that, the problem was also regarded from a mathematical point of view. This is not a first as mathematical modeling has provided a very powerful tool for investigating the dynamics of infectious diseases and controlling them. Previous studies have introduced different models allowing to predict and assess intervention strategies during pandemic spread [7, 8] such as Ebola [9], Tuberculosis [4] or the current Covid-19 [11, 12]. In our case, we consider an SIRD model where we introduce the disease-caused death equation into the model dynamics as our focus, in the second part of this study, is on minimising the number of these deaths. The study is effected over an eight month period as it is the vaccine-induced immunity time interval. During that time, the population is supposed to be completely vaccinated. Thus, the ultimate goal of this study is to minimize the number of deaths among a vaccinated population with basic strategies: mask wear and screening at a minimal cost. This presents the possibility of containing the disease without any extreme measures such as lockdown. Modeling such a situation represents a very good opportunity as it gives a larger view of the situation offering the chance of setting a good vaccination schedule that can lead to a total containment of the disease. In this work, both mathematical and numerical analysis of a controlled epidemiological model of four sub-populations: susceptible, infectious, recovered and dead are presented. Section 2 is a study of the dynamics of the SIR model, its equilibria and their stability. Section 3 focuses on the optimal control problem that aims to reduce the number of the deceased keeping a minimal screening cost. Section 4 is dedicated to the numerical simulations and the discussion. Then, a conclusion was drawn in the last section.

II MODEL DESCRIPTION AND ANALYSIS

This section outlines the formulation of a deterministic SIRD model for COVID-19. The total population at time t is divided into four sub-populations: Susceptible, $S(t)$; Infectious, $I(t)$; Recovered, $R(t)$ and Dead, $D(t)$. Two types of control $u_1(t)$ and $u_2(t)$ are used where $1-u_1(t)$ is the probability of mask wear and $u_2(t)$ is the screening rate. In the Susceptible compartment, $S(t)$, people are recruited into the population at a constant rate, Λ, through migration/birth. They exit this compartment either through infection induced by the disease with the force of infection, $u_1\,\beta\,I(t)$ or natural mortality. The infectious compartment, $I(t)$, gains population through infection induced by the disease at the rate of $u_1(t)\,\beta\,S(t)$. A proportion, α, exits this compartment through recovery at a rate $u_2(t)+\delta$ after screening or end of incubation period, the remaining proportion, $1-\alpha$, of the infectious individuals leaves this compartment at a rate $u_2(t)+\delta$ towards the dead compartment through disease induced death, $D(t)$. Recovered individuals are assumed to develop immunity to COVID-19, and compartments, S , I and R are assumed to have a natural mortality rate, μ. Therefore, the epidemic model is given by the following system:

$$\begin{cases} \frac{dS(t)}{dt} = \Lambda - u_1(t)\beta S(t)I(t) - \mu S(t) \\ \frac{dI(t)}{dt} = u_1(t)\beta S(t)I(t) - (u_2(t) + \mu + \delta)I(t) \\ \frac{dR(t)}{dt} = \alpha\,(u_2(t) + \delta)\,I(t) - \mu R(t) \\ \frac{dD(t)}{dt} = (1-\alpha)\,(u_2(t) + \delta)\,I(t) \end{cases} \tag{1}$$

subject to the following initial conditions

$$S(0) \geq 0, I(0) \geq 0, R(0) \geq 0, D(0) \geq 0$$

The whole population is assumed to be vaccinated and all parameters of the model are positive.

In what follows, we will study the dynamic of the sub-model, susceptible, infected and recovered (SIR) model, in the case where controls are constants.

2.1 Analysis of the SIR model with constant controls

The SIR model corresponds to the first three equations of the system (1):

$$
\begin{cases}
\frac{dS(t)}{dt} = \Lambda - u_1\beta S(t)I(t) - \mu S(t) \\
\frac{dI(t)}{dt} = u_1\beta S(t)I(t) - (u_2 + \mu + \delta)I(t) \\
\frac{dR(t)}{dt} = \alpha\,(u_2 + \delta)\,I(t) - \mu R(t)
\end{cases}
\tag{2}
$$

We aim here to understand the impact of time independent control parameters, i.e., $u_1(t) = u_1$ and $u_2(t) = u_2$, on the transmission dynamics of the COVID-19.

By the following, we prove that the solutions are uniformly bounded in a positive invariant region,

$$
\Omega = \{(S, I, R) \in \mathbb{R}^3_+ : S + I + R \leq \frac{\Lambda}{\mu}\}
\tag{3}
$$

Theorem 1:
For any non-negative initial condition, the solution of system (2) remains non-negative and positively bounded. In addition, the set Ω is positively invariant for the epidemic model (2).

Existence and global stability of equilibrium points

In this section, the existence and the stability of both the disease-free and the endemic equilibria of model (2) are examined.

First, we need to define the basic reproduction number, R_0. This quantity predicts the spread of a disease in the population. It is defined as the average number of secondary infections generated when an infected person is introduced into a host population where everyone is susceptible and it is given by :

$$
R_0 = \frac{\partial_I F(S, I, R)}{\partial_I V(S, I, R)}\Big|_{(\frac{\Lambda}{\mu},0,0)} = \frac{u_1\beta\Lambda}{\mu\,(u_2 + \mu + \delta)}
\tag{4}
$$

where $F(S, I, R) = u_1(t)\beta S(t)I(t)$ and $V(S, I, R) = (u_2(t) + \mu + \delta)I(t)$ denote respectively the rates of the transfer in and out of the infected compartment.

Then, It is easy to show that the system (2) has two steady states: a disease-free equilibrium (DFE) given by $E_0^* = (\frac{\Lambda}{\mu}, 0, 0)$ that exists for any value of the parameters and an endemic equilibrium $E_1^* = (S^*, I^*, R^*)$ in the interior of Ω that exists if and only if $R_0 > 1$ and where,

$$
S^* = \frac{\Lambda}{\mu R_0}, I^* = \frac{\Lambda}{u_2 + \mu + \delta}\left[1 - \frac{1}{R_0}\right], R^* = \frac{\alpha(u_2 + \delta)(R_0 - 1)}{u_1\beta}.
$$

For the global stability of equilibrium we use popular types of Lyapunov functions i.e, the common quadratic and Volterra-type functions.

Theorem 2:
If $R_0 \leq 1$, then the DFE, E_0^*, is globally asymptotically stable on Ω. If $R_0 > 1$, then the endemic equilibrium, E_1^*, is globally asymptotically stable.

III OPTIMAL CONTROL

In this section, we aim to reduce the number of deceased individuals keeping a minimal cost of screening.The cost of mask advertising campaign is treated in two cases: constant in the first part and variable in the second. Note that the controls in this section are no longer considered constant.

3.1 Constant mask wear cost

As aforementioned, the objective is to reduce the number of deceased individuals at a finite time, $D(t^f)$, with a minimal cost of screening $\int_0^{tf} u_2^2(t)dt$.The constant cost of mask advertising campaign has no effect on the objective function,and is assumed equal to zero. Therefore, the objective function that we seek to minimize over a finite time horizon $[0, t^f]$ is given by:

$$\begin{aligned} J(u_1, u_2) &= A_1 D(t^f) + A_2 \int_0^{tf} u_2^2(t)dt \\ &= \int_0^{tf} A_1 \left(1 - \alpha\right) \left(u_2(t) + \delta\right) I(t) + A_2 u_2^2(t)dt \end{aligned} \tag{5}$$

Where the set of admissible controls U is given by

$$U = \{u = (u_1, u_2) \in \left(L^\infty(0; t_f)\right)^2 \mid 0 \leq u_i^{min} \leq u_i(t) \leq u_i^{max} \leq 1 \text{, for } i = 1, 2\}$$

Theorem 3:
There exists an optimal control u^* and a corresponding state variables vector (S^0, I^0, R^0, D^0) that minimizes the objective function.

Theorem 4:
Given optimal controls $u_1^*(t)$, $u_2^*(t)$ and the corresponding solution $S^0(t)$, $I^0(t)$, $R^0(t)$ and $D^0(t)$ of the corresponding state system (1) - (5), there exists adjoint variables λ_1 , λ_2 , λ_3 and λ_4 that satisfy

$$\begin{aligned} \dot{\lambda}_1 &= \beta I(\lambda_1 - \lambda_2)u_1 + \lambda_1 \mu \\ \dot{\lambda}_2 &= \beta S(\lambda_1 - \lambda_2)u_1 + \lambda_2(u_2 + \mu + \delta) - A_1(1 - \alpha)(u_2 + \delta) \\ \dot{\lambda}_3 &= \mu \lambda_3 \\ \dot{\lambda}_4 &= 0 \end{aligned} \tag{6}$$

with transversality conditions:

$$\lambda_i(t_f) = 0\,, \quad i = 1,\, 2,\, 3,\, 4. \tag{7}$$

Furthermore, the optimal control is given by $u^* = (u_1^*, u_2^*)$ where

$$u_1^* = \begin{cases} u_1^{min} & \text{, if } \lambda_2 - \lambda_1 > 0 \\ u_1^{max} & \text{, if } \lambda_2 - \lambda_1 < 0 \end{cases}$$

$$u_2^* = \begin{cases} \frac{(\lambda_2 - (1-\alpha)A_1)I}{2A_2} & \text{, if } u_2^{min} < \frac{(\lambda_2 - (1-\alpha)A_1)I}{2A_2} < u_2^{max} \\ u_2^{min} & \text{, if } \frac{(\lambda_2 - (1-\alpha)A_1)I}{2A_2} < u_2^{min} \\ u_2^{max} & \text{, if } \frac{(\lambda_2 - (1-\alpha)A_1)I}{2A_2} > u_2^{max} \end{cases}$$

3.2 Variable mask cost

In this section we add the mask advertising campaign cost as a quadratic term $-\int_0^{t_f} A_3 u_1(t)$ to the previous objective function. The latter becomes:

$$J(u_1, u_2) = \int_0^{tf} A_1\,(1-\alpha)\,(u_2(t)+\delta)I(t) + A_2 u_2^2(t) - A_3 u_1^2(t)\,dt \tag{8}$$

For the same set of admissible controls aforementioned, one has the following results:

Theorem 5:
There exists an optimal control u^* and a corresponding state variables vector (S^0, I^0, R^0, D^0) that minimizes the objective function (8) if and only if $\frac{A_3}{A_2} \leq (\dfrac{u_2 - v_2}{u_1 - v_1})$.

Theorem 6:
Given optimal controls $u_1^*(t)$, $u_2^*(t)$ and the corresponding solution $S^0(t)$, $I^0(t)$, $R^0(t)$ and $D^0(t)$ of the corresponding state system (1) - (8), there exists adjoint variables λ_1 , λ_2 , λ_3 and λ_4 that satisfy

$$\begin{array}{rcl} \dot{\lambda}_1 & = & \beta I(\lambda_1 - \lambda_2)u_1 + \lambda_1\mu \\ \dot{\lambda}_2 & = & \beta S(\lambda_1 - \lambda_2)u_1 + \lambda_2(u_2 + \mu + \delta) - A_1(1-\alpha)(u_2 + \delta) \\ \dot{\lambda}_3 & = & \mu\lambda_3 \\ \dot{\lambda}_4 & = & 0 \end{array} \tag{9}$$

with transversality conditions:

$$\lambda_i(t_f) = 0\,, \quad i = 1,\,2,\,3,\,4. \tag{10}$$

Furthermore, the optimal control is given by $u^* = (u_1^*, u_2^*)$ where

$$u_1^* = \begin{cases} u_1^{min} & ,\text{if } u_1^{min} < \frac{\beta SI(\lambda_2-\lambda_1)}{2A_3} < 1 \text{ and } H(u_1^{min}) < H(u_1^{max}) \\ u_1^{max} & ,\text{if } 0 < \frac{\beta SI(\lambda_2-\lambda_1)}{2A_3} < u_1^{max} \text{ and } H(u_1^{min}) > H(u_1^{max}) \end{cases}$$

$$u_2^* = \begin{cases} \frac{(\lambda_2-(1-\alpha)A_1)I}{2A_2} & ,\text{if } u_2^{min} < \frac{(\lambda_2-(1-\alpha)A_1)I}{2A_2} < u_2^{max} \\ u_2^{min} & ,\text{if } \frac{(\lambda_2-(1-\alpha)A_1)I}{2A_2} < u_2^{min} \\ u_2^{max} & ,\text{if } \frac{(\lambda_2-(1-\alpha)A_1)I}{2A_2} > u_2^{max} \end{cases}$$

IV NUMERICAL SIMULATIONS AND DISCUSSION

In this section, the system (1) is solved numerically for the constant mask wear case, and the results obtained are presented below. The numerical simulations were carried out by implementing a 4th order Runge-Kutta Method (see, for example [5]). This iterative method consists in solving the system of equation (1). Details of the application of this method are developed in [10]. The parameters used are presented in the table 1. To start, the system is solved using the set of parameters listed above and the following initial conditions

$$[S(0) = 11718548;\ I(0) = 2629;\ R(0) = 0;\ D(0) = 0]$$

Parameters	Description	Values	References
α	The rate at which infected individuals become cured	≈ 0.99	[14]
$N(0)$	The total size of the population	11172177	[15]
β	The disease transmission coefficient	$0.24032955/N(0)$	Fitted
$1/\delta$	The mean duration of infection	5.073 days	Fitted
μ	The death rate	0.000017534	[15]
Λ	The birth rate	510.5937	[15]
A_1	The balancing factor associated to the cost component	30	Assumed
A_2	The balancing factor associated to the cost component	10	Assumed
$1-u_1$	Mask wear rate per unit of time	$0.4 < u_1 < 1$	Assumed
u_2	Screening rate per unit of time	$0 < u_2 < 0.2$	Assumed

Table 1: Description and values of the parameters.

We introduced the control and solved the optimality system. With the use of these parameters, and the adjoint variables dynamics, the following solutions for λ_1 and λ_2 were obtained. For this set of parameters, $\lambda_2 - \lambda_1$ is always positive (see figure 1). According to the optimal control study conducted above, this results in

$$u_1^* = u_1^{min}.$$

For that value of u_1, one has maximal constant mask wear while the screening rate starting at 0.2 remains constant during the first 150 days then decreases to 0 (see figure 2).

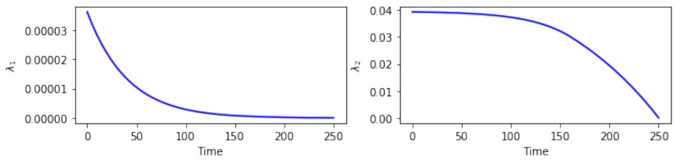

Figure 1: Adjoint variables λ_1 and λ_2

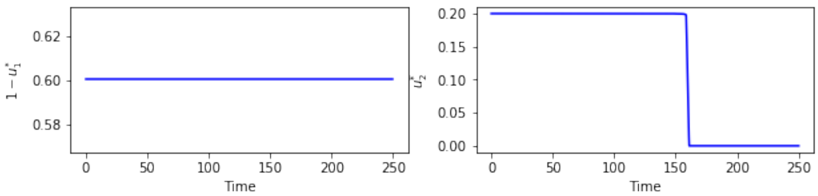

Figure 2: Optimal mask wear rate $1 - u_1^*$ (left) and screening rate u_2^* (right) per unit of time

Then, the state variables were plotted in two cases: controlled and uncontrolled. In the absence of any form of control, the susceptible curve starts decreasing almost from the start until it reaches a value near zero. On the other hand, the curve of the infected reaches a peak that exceeds $2.\,10^5$ and the number of deaths reaches $4.\,10^4$. However, once the system is controlled, a huge difference in the dynamics is observed. The susceptible number is increasing starting from day 100 as opposed to the the infections that start at a maximal value of $3.\,10^4$ then decrease to 0. The dead curve is still increasing; however, to a maximal value less than 450 (see figure 3).

The coefficients, A_1 and A_2, are balancing cost factors. We assume that A_1 associated with the number of deaths $D(t^f)$ is greater than or equal to A_2, associated with the screening u_2. The fractions of the weighing factors, $A_1/A_2 = 1,\ 3,\ 10$ and 100, are presented in Figure (4). And to illustrate the optimal strategy we have chosen the weighing factor, $A1/A2 = 3$ since the only change observed was in the values of the controls rather than the dynamics.

In order to present the importance of maximal mask wear and screening values, the state variables were represented for two values of maximal screening $u_2^{max} \in \{0.2;\, 0.5\}$ and four different values of mask wear $0.2,\ 0.4,\ 0.6,\ 0.9$ (See figure 5). In both cases, the same behaviour is

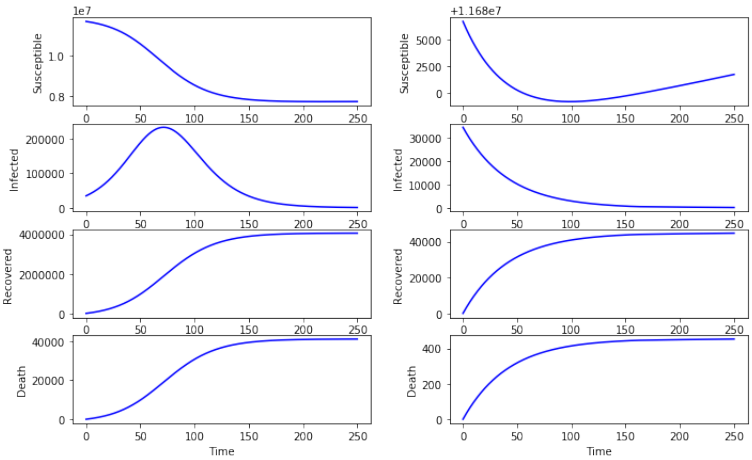

Figure 3: Dynamics of state variables per unit of time in two cases: without any control measures $1 - u_1 = u_2 = 0$ (a) and with optimal control pair (u_1^*, u_2^*) (b)

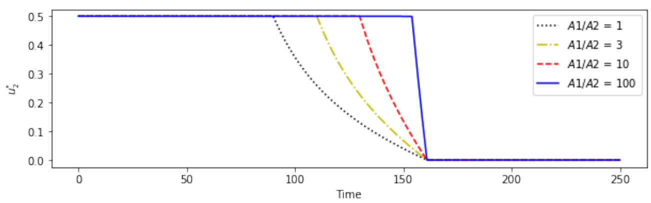

Figure 4: The screening rate u_2^* per unit of time

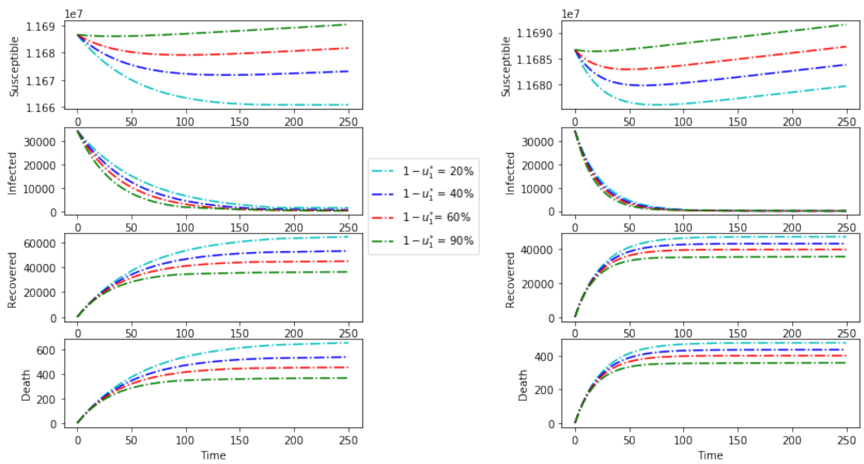

Figure 5: Simulation of SIRD model for two values of maximal screening $u_2^{max} \in \{0.2, 0.5\}$ and four different values of mask wear $1 - u_1^{min} \in \{0.2, 0.4, 0.6, 0.9\}$

observed: infections decrease, both the susceptible and recovered increase and and deaths had a threshold. However, a difference of the pace is noticed as infections decrease much faster for high screening values and deaths maximal value is lower: less than 450 with high screening as opposed to over 600 for low screening.

A ANNEX 1

Proof. The existence of the optimal control pair can be obtained using a result from [5, 8].
In fact, one can easily verify that:
1. The set of controls and corresponding state variables is nonempty.
2. The admissible set U is convex and closed.

3. The right hand side of the state system 1 is bounded by a linear function in the state and control variables.
4. The integrand of the objective functional L is convex on U and there exists constants $\omega_1 > 0, \omega_2 > 0$ and $\rho > 1$ such that

$$L(u) \geq \omega_2 + \omega_1(|u_1|^2 + |u_2|^2)^{\frac{\rho}{2}}.$$

□

Proof. In order to determine the optimal control, Pontryagin's Minimum Principle was used [5]. The latter changes the optimality system into a study of the Hamiltonian variations through the use of adjoint functions . The Hamiltonian is given by

$$H(t, u, X, \lambda) =< \lambda(t), \dot{X}(t) > +A_1\,(1-\alpha)\,(u_2(t)+\delta)\,I(t) + A_2 u_2^2(t)$$

where $X = (S, I, R, D)$ is the vector of state variables and $\lambda = (\lambda_1(t), \lambda_2(t), \lambda_3(t), \lambda_4(t))$ is the vector of adjoint variables and $< .,. >$ is the scalar product. According to Pontryagin's minimum principle, the adjoint functions $(\lambda_1(t), \lambda_2(t), \lambda_3(t), \lambda_4(t))$ have the following dynamics

$$\begin{array}{rcl}
\dot{\lambda}_1 &=& \beta I(\lambda_1 - \lambda_2)u_1 + \lambda_1\mu \\
\dot{\lambda}_2 &=& \beta S(\lambda_1 - \lambda_2)u_1 + \lambda_2(u_2 + \mu + \delta) - \lambda_3\alpha(u_2+\delta) - (1-\alpha)(u_2+\delta)\lambda_4 - A_1(1-\alpha)(u_2+\delta) \\
\dot{\lambda}_3 &=& \mu\lambda_3 \\
\dot{\lambda}_4 &=& 0
\end{array}$$

with the final conditions

$$\lambda(t^f) = (0, 0, 0, 0).$$

From the third and fourth equations we can deduce that $\lambda_3 \equiv 0$ and $\lambda_4 \equiv 0$.
Consequently, the Hamiltonian becomes

$$\begin{array}{rcl}
H &=& (\Lambda - u_1(t)\beta S(t)I(t) - \mu S(t))\,\lambda_1 + (u_1(t)\beta S(t)I(t) - (u_2(t)+\mu + \delta)I(t))\,\lambda_2 \\
&& +A_1\,(1\;\alpha)\,(u_2(t) + \delta)I(t) + A_2 u_2^2(t).
\end{array} \tag{11}$$

and the adjoint variables dynamics is reduced to

$$\begin{array}{rcl}
\dot{\lambda}_1 &=& \beta I(\lambda_1 - \lambda_2)u_1 + \lambda_1\mu \\
\dot{\lambda}_2 &=& \beta S(\lambda_1 - \lambda_2)u_1 + \lambda_2(u_2 + \mu + \delta) - A_1(1-\alpha)(u_2+\delta)
\end{array}$$

Also, the Pontryagin's Minimum Principle states that the optimal control u^* minimizes the Hamiltonian, hence we should seek the minimum of H . So we need to study the critical points of the Hamiltonian.

$$\left\{\begin{array}{rcl}
\frac{\partial H}{\partial u_1} &=& \beta SI(\lambda_2 - \lambda_1) \\
\frac{\partial H}{\partial u_2} &=& (-\lambda_2 + (1-\alpha)A_1)\,I + 2A_2u_2
\end{array}\right.$$

The equation $\frac{\partial H}{\partial u_2} = 0$ implies that

$$u_2^* = \frac{(\lambda_2 - (1-\alpha)A_1)\,I}{2A_2}$$

whereas the first equation shows that the minimum is either reached at $u_1^* = u_1^{min}$ or $u_1^* = u_1^{max}$ according to the sign of $\lambda_2 - \lambda_1$.

In fact when u_1 is supposed constant; H would depend on u_2 only and therefore u_2^* is a minimum to H since $A_2 > 0$. In that case, one has

$$H(u_1, u_2) > H(u_1, u_2^*)$$

Since $u_1^{min} \leq u_1 \leq u_1^{max}$ then two scenarios are possible

- If $\beta SI(\lambda_2 - \lambda_1) > 0$ i.e. $\lambda_2 - \lambda_1 > 0$ then

$$(\lambda_2 - \lambda_1)u_1^{min} \leq (\lambda_2 - \lambda_1)u_1 \leq (\lambda_2 - \lambda_1)u_1^{max}$$

  and consequently,

$$H(u_1, u_2) \geq H(u_1, u_2^*) \geq H(u_1^{min}, u_2^*)$$

- If $\beta SI(\lambda_2-\lambda_1)<0$ i.e. $\lambda_2-\lambda_1<0$ then

$$(\lambda_2-\lambda_1)u_1^{min}\geq(\lambda_2-\lambda_1)u_1\geq(\lambda_2-\lambda_1)u_1^{max}$$

and consequently,

$$H(u_1,u_2)\geq H(u_1,u_2^*)\geq H(u_1^{max},u_2^*)$$

Note that u_2^* must satisfy $u_2^{min}<u_2^*<u_2^{max}$ to be taken into consideration. Otherwise,

- $\min\limits_{u_2\in[u_2^{min},u_2^{max}]} H=H(u_2^{min})$ if $\frac{\partial H}{\partial u_2}>0$ i.e. $(-\lambda_2+(1-\alpha)A_1)\,I+2A_2u_2>0$
- $\min\limits_{u_2\in[u_2^{min},u_2^{max}]} H=H(u_2^{max})$ if $\frac{\partial H}{\partial u_2}<0$ i.e. $(-\lambda_2+(1-\alpha)A_1)\,I+2A_2u_2<0$.

Assume now that there exists a subset $[t_0,t_1]\in[0,t_f]$ such that $\frac{\partial H}{\partial u}=0$ for all $t\in[t_0,t_1]$. This implies that

$$\begin{cases}\beta SI(\lambda_2-\lambda_1) &= 0\\ (-\lambda_2+(1-\alpha)A_1)\,I+2A_2u_2 &= 0\end{cases}$$

And consequently

$$\begin{cases}\beta SI(\lambda_2-\lambda_1) &= 0\\ (-\lambda_2+(1-\alpha)A_1)\,I &= 0\\ A_2 &= 0\end{cases}$$

Since $A_2>0$, we deduce that it is not possible to have $\frac{\partial H}{\partial u_2}=0$ and therefore we cannot discuss the case of singular control in the usual terms. However, it is possible to have $\frac{\partial H}{\partial u_1}=0$ which implies that $\beta SI(\lambda_2-\lambda_1)=0$. Consequently, either $S.I=0$ or $\lambda_2-\lambda_1=0$. As the first case does not present quite an interesting case of study, we move to the latter that yields

$$\begin{aligned} & \lambda_1 &= \lambda_2\\ \Rightarrow\quad & \dot{\lambda_1} &= \dot{\lambda_2}\\ \Rightarrow\quad & (u_2+\delta)(\lambda_1-A_1(1-\alpha)) &= 0\end{aligned}$$

Thus, either $u_2=-\delta$ which is not taken into account since $-\delta\notin[0;u_2^*]$ or $\lambda_1=A_1(1-\alpha)=\lambda_2$. However, according to the co-state variables dynamics, one has $\dot{\lambda_1}=\mu\lambda_1$ which implies that $\lambda_1(t)=\lambda_1(t_0)e^{\mu(t-t_0)}$.Consequently, $\lambda_1(t_0)e^{\mu(t-t_0)}=A_1(1-\alpha)$, $\forall t\in[t_0,t_1]$. This equality is absurd except for one particular case $\alpha=1$ and $\lambda_1(t_0)=0$. Therefore, the existence of an interval $[t_0;t_1]$ such that $\frac{\partial H}{\partial u}=0\ \forall t\in[t_0;t_1]$ is not possible. □

Proof. The two first conditions for the existence of optimal control are checked in the first case and remain unchanged.
Now , let $f(u)=A_1\,(1-\alpha)\,(u_2+\delta)I+A_2u_2^2-A_3u_1^2$ and $u,v\in U$

$$f(u)-f(v)\quad=\quad(u_1-v_1)f'_{u_1}+(u_2-v_2)f'_{u_2}-A_2(u_2-v_2)^2+A_3(u_1-v_1)^2$$

$$f(u)-f(v)\leq(u_1-v_1)f'_{u_1}+(u_2-v_2)f'_{u_2}\iff\frac{A_3}{A_2}\leq(\frac{u_2-v_2}{u_1-v_1})^2$$

Moreover, one has

$$f(u)\geq\omega_1(|u_1|^2+|u_2|^2)^{\frac{2}{2}}+\omega_2$$

where $\omega_1=A_2$ and $\omega_2<A_1(1-\alpha)(u_2+\delta)-(A_3+A_2)\,u_1^2$ □

Proof. Using Pontryagin's minimum principle, one has the following dynamics for the adjoint functions $(\lambda_1(t),\lambda_2(t),\lambda_3(t),\lambda_4(t))$

$$\begin{aligned}\dot{\lambda_1} &= \beta I(\lambda_1-\lambda_2)u_1+\lambda_1\mu\\ \dot{\lambda_2} &= \beta S(\lambda_1-\lambda_2)u_1+\lambda_2(u_2+\mu+\delta)-\lambda_3\alpha(u_2+\delta)-(1-\alpha)(u_2+\delta)\lambda_4-A_1(1-\alpha)(u_2+\delta)\\ \dot{\lambda_3} &= \mu\lambda_3\\ \dot{\lambda_4} &= 0\end{aligned}$$

with the final conditions

$$\lambda(t^f)=(0,0,0,0).$$

Similarly to the first case, one can deduce from the third and fourth equations that $\lambda_3(t) = 0$ and $\lambda_4(t) = 0$. Consequently, the Hamiltonian becomes

$$\begin{aligned} H &= (\Lambda - u_1(t)\beta S(t)I(t) - \mu S(t))\,\lambda_1 + (u_1(t)\beta S(t)I(t) - (u_2(t) + \mu + \delta)I(t))\,\lambda_2 \\ &\quad + A_1\,(1\ \alpha)\,(u_2(t) + \delta)I(t) + A_2 u_2^2(t). \end{aligned} \tag{12}$$

and the adjoint variables dynamics is reduced to

$$\begin{aligned} \dot{\lambda}_1 &= \beta I(\lambda_1 - \lambda_2)u_1 + \lambda_1\mu \\ \dot{\lambda}_2 &= \beta S(\lambda_1 - \lambda_2)u_1 + \lambda_2(u_2 + \mu + \delta) - A_1(1-\alpha)(u_2 + \delta) \\ \dot{\lambda}_3 &= \mu\lambda_3 \\ \dot{\lambda}_4 &= 0 \end{aligned}$$

Also, the Pontryagin's Minimum Principle states that the optimal control u^* minimizes the Hamiltonian, hence we should seek the minimum of H . So we need to study the critical points of the Hamiltonian. A critical point of H, $u^* = (u_1^*, u_2^*)$ satisfies $\frac{dH}{du} = 0$ where

$$\begin{cases} \frac{\partial H}{\partial u_1} &= \beta SI(\lambda_2 - \lambda_1) - 2A_3 u_1 \\ \frac{\partial H}{\partial u_2} &= (-\lambda_2 + (1-\alpha)A_1)\,I + 2A_2 u_2 \end{cases}$$

The equation $\frac{\partial H}{\partial u_2} = 0$ implies that

$$u_2^* = \frac{(\lambda_2 - (1-\alpha)A_1)\,I}{2A_2}$$

The first equation, shows that the critical value is $u_1^{crit} = \frac{\beta SI(\lambda_2 - \lambda_1)}{2A_3}$. However, it is easy to verify that this value is a maximum to the parabola $H(u_2)$. In fact, one has three possible cases

- If $u_1^{crit} \in [u_1^{min}; u_1^{max}]$, then the minimum is u_1^{min} if $H(u_1^{min}) < H(u_1^{max})$ and u_1^{max} otherwise.
- If $u_1^{crit} > u_1^{max}$, then the minimum is reached at u_1^{min}.
- If $u_1^{crit} < u_1^{min}$, then the minimum is reached at u_1^{max}.

□

B ANNEX 2

Parameter estimation

The root mean square error (RMSE) [6] is a frequently used method to measure the difference between the values predicted by a model and the values observed in reality. Let X_{obs} be the vector of the observed values and X_{model} the vector of modeled ones. The RMSE of a prediction model with respect to the estimated variable X_{model} is defined as follows

$$RMSE = \sqrt{\frac{1}{n}\sum_{j=1}^{n}\left(X_{model,j} - X_{obs,j}\right)^2}$$

Hence, to obtain optimal parameters $\{\beta,\ \delta\}$ for our model, one should solve the following problem :

$$min\ RMSE$$

Here, the fit is measured by computing the value of the RMSE function using data of deaths for the beginning of the second wave in Tunisia which is calibrated from June 2021, provided by [1] as X_{obs} data. X_{model} is the death data obtained by the SIRD model (1) subject to the following initial condition

$$S(0) = 1686692,\ I(0) = 34485,\ R(0) = 0,\ D(0) = 0$$

In addition, to minimize the RMSE function, we used the genetic method [2] to update the parameters β and δ. Figure 6 shows the result of the fitted values using the optimal parameters β and δ.

[1]https://covid19.who.int/WHO-COVID-19-global-data.csv
[2]https://github.com/rmsolgi/geneticalgorithm

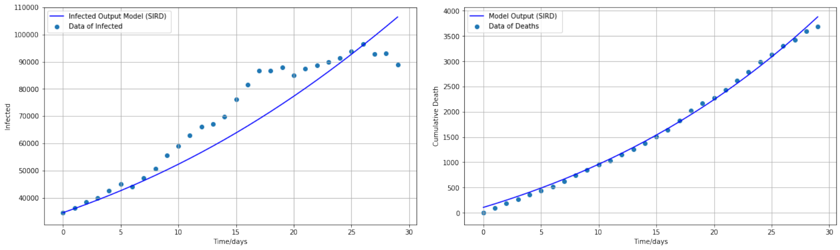

Figure 6: The fitted value using the optimal parameters $\beta = 1.3201448967113115e - 08$ and $\delta = 0.098575$

C CONCLUSION

In this work, a study of COVID-19 transmission for the case of Tunisia was carried out. A four compartmental mathematical model of a vaccinated population with mask wear and screening as time-dependent control measures is developed. The model is proven to have an invariant region where it is well-posed and makes biological sense to be studied for human population. Different properties of the model including global stability analysis of the equilibria have been studied. Some of the parameter estimates were taken from literature and the remaining parameters were estimated based on real daily data of COVID-19 confirmed cases of Tunisia. An optimal analysis of the model for the purpose of assessing the effect of mask wear by the individuals and screening companions was conducted. The results showed that the optimal practice of combination of these two strategies in a vaccinated population significantly reduces the number of infections and deaths. And for quicker results it is required to set higher maximal values of screening and mask wear (see figure 5).

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REFERENCES

Publications

[1] J. P. LaSalle. "Some extensions of Lyapunov's second method". In: *IRE Transactions Circuit Theory* (1960).

[2] J. P. LaSalle. *The stability of Dynamical Systems*. Philadelphia: SIAM, 1976.

[3] M. Vidyasagar. "Decomposition techniques for large-scale systems with non additive interactions : Stability and stabilizability". In: *IEEE Transactions on Automatic Control* (1980).

[4] E. Jung, S. Lenhart, and Z. Feng. "Optimal control of treatment in a two-strain Tuberculosis model". In: *Discrete and continuous dynamical systems–seriesB* (2002).

[5] S. Lenhart and J. Workman. *Optimal Control Applied to Biological Models*. London: Chapman and Hall, 2007.

[6] A. J. Haug. *Bayesian estimation and tracking: a practical guide*. New Jersy: John Wiley and Sons, 2012.

[7] E. Laarabi H.and Labriji, M. Rachik, and A. Kaddarb. "Optimal control of an epidemic model with a saturated incidence rate". In: *Nonlinear Analysis: Modelling and Control* (2012).

[8] H. Laarabi, A. Abta, and K. Hattaf. "Optimal Control of a Delayed SIRS Epidemic Model with Vaccination and Treatment". In: *Acta Biotheor* (2015).

[9] H. Boujakjian. “Modeling the spread of Ebola with SEIR and optimal control”. In: *SIAM* (2016).

[10] C. Campos C.and Silva and D. Torres. “Numerical Optimal Control of HIV Transmission in Octave/MATLAB”. In: *Numerical and Symbolic Computation: Developments and Applications* (2019).

[11] E. N. Grigorieva E. V.and Khailov and A. Torres. “Optimal quarantine strategies for covid-19 control models”. In: *arXiv* (2020).

[12] A. Mallela. “Optimal Control applied to a SEIR model of 2019-nCoV with social distancing”. In: *medRxiv* (2020).

[13] S. Ben Miled and A. Kebir. “Simulations of the spread of COVID-19 and control policies in Tunisia”. In: *Journal of Public Health in Africa* (2021).

[14] W. H. O. (WHO). “Novel Coronavirus (2019-nCoV) situation reports”. In: ().

[15] CIA. “Indicateurs du World-Factbook [archive] publié par la CIA”. In: *archive* ().