# Shear shallow water modeling of sediments transport flows.

**A. R. Ngatcha Ndengna[1,*], B. Nkonga[2], A. Njifenjou[1,3], R. Onguene[4]**

[1]Laboratory E3M, National Higher Polytechnic School of Douala, University of Douala , Cameroon
[2] Université Côte d'Azur & Inria/CASTOR/AmFoDuc, CNRS, LJAD, Nice Sophia-Antipolis, France
[3]Dep. of Mech. Engineering, National Advanced School of Engineering, University of Yaounde I, Cameroon
[4]Laboratory of Technology and Applied Science, University of Douala, Cameroon

*E-mail : Arno Ngatcha.arnongatcha@gmail.com

### Abstract

The classical shallow water model of sediment transport describes the hydro-morphodynamic process without horizontal velocity shear along the vertical. Nevertheless, for the coastal flows we are interested in, it seems important to take into account these shear effects. In this paper, we develop a new sediment transport model incorporating both the shear velocity along the vertical and the spatial variation of the mixing density. The starting point is the 2D equations for the evolution of mixing quantities and sediment volume rate. These equations describe the evolution of fluid mixing in a domain bounded by a dynamic water surface and water bed. Taking into account the kinematic conditions on the moving surfaces, we apply an average along the depth of the equations to obtain simplified equations. Contrary to the classical sediment transport model, the second order vertical fluctuations of the horizontal velocity are considered. Taking into account the kinematic conditions on the moving surfaces, we apply an average along the depth of the equations to obtain simplified equations. An evolution equation is formulated for this quantity involved in the dynamics of the mean velocities as well as those of the sediment volume fraction and the bed position. The resulting model has a wider range of validity and integrates the morphodynamic processes proposed in the literature. The proposed derivation is in the context of recent developments with the additional presence of sediment and a dynamic bed.

### Keywords

Shear shallow water; Sediment transport, Morphological dynamic; Douala Coastal city.

## I INTRODUCTION

This coastal city is subject to very heavy rains that frequently cause flooding and lead to the transport of sediments and their deposit in the port channel, which hinders the optimal operation of the port of Douala. The flooding result from the interactions between oceanic tidal forcing, torrential rains with a constrained runoff, transport of sediments of variable characteristics : either mineral (sand, clays,...) or organic (plants, sewage residues,...). The currents here are very strong and result in very important shear effects which are essential in the realistic description of the dynamics of these flows. Mathematical models describing such flows often assume that there is a carrier fluid (water) and that the sediments are either suspended in the carrier fluid (Suspended-Layer) or constitute a bottom layer in motion or at rest (Bedload-Layer).

The interactions between these two main layers are done through a third exchange layer, that is in general so thin that its thickness can be neglected. In the suspended-layer, both the main fluid and the suspended particles can be assumed to be incompressible. Instead of following the individual evolution of each suspended particle in the fluid, many models adopt a continuum scale formulation of the particles that can be reduced to their volume concentration in the carrier fluid.

Therefore, the father-model considered here is a 2D Multi-fluid formulation of the fluid-sediment mixture described as [**van_Rijn_1987**] (Chap. 2)

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{1}$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho uu + p) + \frac{\partial}{\partial z}(\rho uw) = \mathbf{F}_x \tag{2}$$

$$\frac{\partial}{\partial t}(\rho w) + \frac{\partial}{\partial x}(\rho uw) + \frac{\partial}{\partial z}(\rho ww + p) = \mathbf{F}_z \tag{3}$$

$$\frac{\partial \alpha_s}{\partial t} + \frac{\partial}{\partial x}(u\alpha_s) + \frac{\partial}{\partial z}(w\alpha_s) = \nabla \cdot (\underline{\mathbf{D}}_s \nabla \alpha_s) \tag{4}$$

where $x$ and $z$ are respectively the horizontal and the vertical coordinate. The mixture density is defined as $\rho = \rho_f(1 - \alpha_s) + \rho_s \alpha_s$, where the main fluid density $\rho_f$ and the suspended particles density $\rho_s$ are assumed to be constant in space and time. The components of the mixture velocity are $u$ in the horizontal direction and $w$ in the vertical direction. The external forces, including viscous forces are defined by the vector $\mathbf{F}$. The pressure $p$ will be defined according to the hydrostatic assumption. The the volume fraction of the sediments is denoted by $\alpha_s$. The diffusion contribution in the evolution of the sediments volume fraction $\alpha_s$ is there to take into account the deviations from the mixture velocity of the sediment velocity. The suspension is sufficiently dilute (Boussinesq approximation) to consider that the value of the kinematic viscosity of water/sediment ($\underline{\mathbf{D}}_s$) is equals to the corresponding to clear water ($\underline{\mathbf{D}}_f$): $\underline{\mathbf{D}}_s = \underline{\mathbf{D}}_f$. There is a counterpart in the evolution of the fluid volume fraction that will compensate to achieve the following evolution of the mixture density:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial z}(\rho w) = 0 \tag{5}$$

The partial differential equations (1)-(4) described the dynamic of the flow by the variables $u$, $w$, and $\alpha_s$ in a domain defined by two moving surfaces parameterized by $z = \xi(t, x)$ for the upper surface of the flow and by $z = \mathrm{b}(x) + \mathrm{z}_b(t, x)$ for the bedload layer. For coastal flow, it is well known that the shear stress play an important role in the flow dynamic. This has motivated the use of turbulent models such as $k - \varepsilon$ models for numerical simulations where the turbulent scales are not resolved. This numerical approaches are based on the Reynolds procedure and offers a simple way to take into account the reduction of the mixing coefficients induced by turbulence. On the other hand, simplified modeling are also used by introducing the depth-averaging process that usually neglect the fluctuation of the velocities in the vertical direction [**Simpson_and_Castelltort_2006**]. This paper aims to propose an intermediate modeling where shear process are taken into account in the depth averaging model. This approach is based on recent developments [**Teshukov_2007**, **Gavrilyuk_et_Al_2018**, **Praveen_et_Al_2020**] that take into account the velocity fluctuations in the form of an evolution equation for the Reynolds tensor.

## II   SHEAR SHALLOW WATER MODEL WITH SEDIMENT TRANSPORT.

As we have already point out, the sediment flow is confined in a domain bounded by an upper interface with the atmosphere and a bottom bedload interface. Theses interfaces are evolving in time and their positions are

$$z(t) = \xi(t,x) \qquad \text{and} \qquad z(t) = b(x) + z_b(t,x) \tag{6}$$

where $\xi(t,x)$ and $b(x)$ the vertical positions of respectively the air-fluid interface and the non-erodible bedload (assumed independent of time). The thickness of the deposited sediment is $z_b(t,x)$ and variable in space and time, such that the interface of the fluid with the sedimentary bedload is at the vertical position $z = b(x) + z_b(t,x)$. In order to model the dynamic of the mouving interfaces, let us considered a fluid particle located at one of that surfaces at the time $t$. The position in Lagrangian coordinates is given by $\mathcal{X}(x(t),t) = (x(t), z(t))^T$ where for the upper surface we have $z(t) = \xi(t,x)$. Then the time derivative of the function $F_\xi(t) = z(t) - \xi(t,x(t))$ function will defined the net of particles that are detached and/or agglomerated to the interface. On the other hand the velocity is the time derivative of the Lagrangian coordinates. Therefore, the kinematic condition write as

$$\frac{\partial \xi}{\partial t} + u(t,x,\xi)\frac{\partial \xi}{\partial x} - w(t,x,\xi) = \frac{dF_\xi}{dt} \tag{7}$$

The material derivative $\dfrac{dF_\xi}{dt}$ is the net water volume rate change per unit of time related to evaporation and/or rainfall. This exchange rate is one of the input data of the problem which is generally obtained from the measuring stations covering the study area. In the same way, at the bedload interface, using the fact that the material derivative of $b(x(t))$ is zero, we obtain with $F_b(t) = z(t) - z_b^\star(t,x(t))$, the following kinematic condition

$$\frac{\partial z_b^\star}{\partial t} + u(t,x,z_b^\star)\frac{\partial z_b^\star}{\partial x} - w(t,x,z_b^\star) = \frac{dF_b}{dt} \tag{8}$$

where $z_b^\star(t,x(t)) = b(x(t)) + z_b(t,x(t))$ is the elevation of the bedload interface. The material derivative $\dfrac{dF_b}{dt}$ is now a function of the balance between the particles that are eroded and/or deposited on the surface and of the local bedload porosity $\phi_s$ with $0 \leq \phi_s < 1$. This the morphological change of the bathymetry is the volume rate exchange per unit of time. The volume increment $dF_b$ can be decomposed as the sum of the increment of sediments volume plus a fraction of void (porosity $\phi_s$) filled by the clean water. Moreover, the volume increment of sediment is balance between the amount of sediment left behind by the current (deposed) and the amount of sediment carried away (eroded) from the bedload interface

$$dF_b = d\vartheta_s^{deposed} - d\vartheta_s^{eroded} + \phi_s dF_b$$

Therefore, denoting $\mathcal{D} = \frac{d\vartheta_s}{dt}\big|^{deposed}$ the deposition rate and $\mathcal{E} = \frac{d\vartheta_s}{dt}\big|^{eroded}$ the erosion rate, we obtain

$$\frac{dF_b}{dt} = \frac{\mathcal{D} - \mathcal{E}}{1 - \phi_s}$$

To define the morphological change, we need to give explicit expressions for the erosion $\mathcal{E}$, the deposition $\mathcal{D}$ and the bedload porosity $\phi_s$. Proper modeling of volume rate change is not obvious

and many strategies are available in the literature [**Einstein_1950**, **Luque_and_Beek_1976**, **Yalin_1977**, **Nagakawa_Tsujimoto_1980**, **van_Rijn_1984c** ]. The deposition rate of sediments $\mathcal{D}$ is almost equal to the vertical flux of particle at the boundary [**Fang_and_Rodi_2003** ](page 381):

$$\mathcal{D} \simeq \mathrm{w}_s \alpha_s \left(t, x, \mathrm{z}_b^\star\right) \tag{9}$$

where $\mathrm{w}_s$ is the sediment settling velocity and $\alpha_s \left(t, \mathbf{x}, \mathrm{z}_b^\star\right)$ is the volume concentration of sediment at the vicinity above the bedload [**van_Rijn_1987** ] :

$$\mathrm{w}_s = g \frac{(\rho_s - \rho_f)\mathrm{d}_s^2}{18\mu_f \rho_f}$$

The amount of grains eroded from the bedload per unit area and time, also defined as the pick-up rate, for small particles sediments at low flow velocity, is estimated as [**van_Rijn_1984c** ] and later modified in [**van_Rijn_et_All_2019** ]:

$$\mathcal{E} = \zeta \mathrm{f}_d \rho_s \mathrm{d}_\star^{0.3} \sqrt{g\mathrm{d}_s \left(\frac{\rho_s - \rho_f}{\rho_f}\right)} \left(\max\left(0, \frac{\theta' - \theta_{\mathrm{cr}}}{\theta_{\mathrm{cr}}}\right)\right)^{\frac{3}{2}}$$

where $\zeta = 3.310^{-4}$, $\mathrm{f}_d = \min\left(1, \frac{1}{\theta'}\right)$ a damping factor,

$$\mathrm{d}_\star = \mathrm{d}_s \left(g\frac{\rho_s - \rho_f}{\rho_f \mu_f^2}\right)^{\frac{1}{3}}, \quad \theta' = \frac{\tau_b}{g\mathrm{d}_s(\rho_s - \rho_f)}, \quad \tau_b = \rho_f g \left(\frac{\|\overline{\boldsymbol{u}}\|}{\mathrm{C}_z}\right)^2, \quad \mathrm{C}_z = 5.75\sqrt{g} \ln\left(\frac{12\mathrm{R}}{3\mathrm{d}_l}\right)$$

The erosion pick-up rate $\mathcal{E}$ is given in mass per unit area and time (kg/m$^2$/s), where $\mathrm{d}_\star$ the dimensionless grain size parameter, $\mathrm{d}_s$ median grain size (m), $\mu_f$ the kinematic viscosity coefficient of fluid (m$^2$/s), $\rho_s$ the sediment density (kg/m$^3$), $\rho_f$ the fluid density (kg/m$^3$), $\theta'$ grain-related parameter or particle mobility parameter that is the ratio of the hydrodynamic forces by the submerged particle weight, $\tau_b$ the average grain-related bedload-shear stress (N/m$^2$) due to currents and waves, $\overline{\boldsymbol{u}}$ the depth-averaged flow velocity (m/s), $\mathrm{C}_z$ the Chézy-coefficient ($\sqrt{m}$/s), $\mathrm{R}$ the hydraulic radius (m), (assumed to be equal to water depth), $\theta_{\mathrm{cr}}$ the Shields value at initiation of motion, $g$ the gravity acceleration (m/s$^2$). The sediment particles will leave the interface when the Shields parameter $\theta'$ exceeds the critical value $\theta_{\mathrm{cr}}$.

**Bedload transport.** The bedload transport is difficult to predict because of the mixing of antagonistic flow regimes : fast and slow, the non-equilibrium and noise-driven, temporal and spatial scalings, heterogeneities and nonlinearities. Moreover, there are threshold effects, cascades of interacting processes, hysteresis, poor knowledge of initial and boundary conditions, difficulties in obtaining reliable measurements. A simple transport of sediment can be considered by assuming the bedload layer as a uniform reservoir of independent particles [**Charru_et_Al_2004**, **Mouilleron_et_Al_2009** ]. The mass conservation of moving particles is then applied to formulate the transport at the bedload interface in term of a transport discharge flux $\mathrm{q}_b^\star$ :

$$u\left(t, x, \mathrm{z}_b^\star\right) \frac{\partial \mathrm{z}_b^\star}{\partial x} - w\left(t, x, \mathrm{z}_b^\star\right) \simeq \frac{\partial \mathrm{q}_b^\star}{\partial x} \tag{10}$$

Here the motion at the bedload interface is balanced by the gradient of the horizontal mass sediment flux $\mathrm{q}_b^\star$. Indeed, with an interface normal $\left(\frac{\partial \mathrm{z}_b^\star}{\partial x}, -1\right)$ the left-hand side(LHS) of the

previous equation can be view as an asymptotic limit of a divergence formulation. The evolution of the bedload interface elevation is then given by the flux form of the Exner's equation [**Paola_and_Voller_2005** ](Eq. 15, page 4)

$$\frac{\partial z_b^\star}{\partial t} + \frac{\partial q_b^\star}{\partial x} = \frac{\mathcal{D} - \mathcal{E}}{1 - \phi_s} \tag{11}$$

In the considered context, the sediment transport flux is computed using the energetic-based formula of [**Bailard_1981** ] where small values proportional to the bottom slope are neglected. A similar approximation is also used in [**Liu_et_Al_2015** ] where

$$q_b^\star \simeq \frac{\mu \overline{u^2 u}}{1 - \phi_s} \tag{12}$$

where $\overline{u^2 u}$ is the depth-average of the horizontal speed to the cubic power, $\mu$ is a coefficient usually obtained experimentally by taking into account the grain diameter and the kinematic viscosity of the sediment mixture. In [**Liu_et_Al_2015** ], the depth-average is approximated as $\overline{u^2 u} \simeq \overline{u}^2 \overline{u}$. In the coming sections, we will propose an extended version of this modeling, including shear fluctuations of the velocities in the vertical direction. Nevertheless, given the numerical difficulties (loss of hyperbolicity) encountered with this flux-formulation of the bedload transport, another alternative that is numerically suitable is to formulate the bedload velocities as a functions of the depth-average of the above flow characteristics.

## 2.1 Depth averaged equations

We define the depth average $\overline{\phi} = \overline{\phi}(t, x)$ for any quantity $\phi(t, x, z)$ by

$$\overline{\phi} = \frac{1}{h} \int_{z_b^\star}^{\xi} \phi dz \qquad \text{where} \qquad z_b^\star = b + z_b, \quad h(t, x) = \xi(t, x) - z_b^\star(t, x) \tag{13}$$

The fluctuation with respect to the average value is $\phi' = \phi - \overline{\phi}$ and clearly we have

$$\overline{\phi'} = 0$$

Moreover, we note the following identities

$$\int_{z_b^\star}^{\xi} \frac{\partial \phi}{\partial x} dz = \frac{\partial}{\partial x}\left(h\overline{\phi}\right) - \frac{\partial \xi}{\partial x}\phi(t, x, \xi) + \frac{\partial z_b^\star}{\partial x}\phi(t, x, z_b^\star)$$

and

$$\int_{z_b^\star}^{\xi} \frac{\partial \phi}{\partial t} dz = \frac{\partial}{\partial t}\left(h\overline{\phi}\right) - \frac{\partial \xi}{\partial t}\phi(t, x, \xi) + \frac{\partial z_b^\star}{\partial t}\phi(t, x, z_b^\star)$$

Integrating the divergence free equation over the depth of water and using the previous relations for $\phi \equiv u$ yields

$$\frac{\partial}{\partial x}\left(h\overline{u}\right) - \frac{\partial \xi}{\partial x}u(t, x, \xi) + \frac{\partial z_b^\star}{\partial x}u(t, x, z_b^\star) + w(t, x, \xi) - w(t, x, z_b^\star) = 0$$

Then, using the kinematic condition (7) and (8), we derive that

$$-\frac{\partial \xi}{\partial x} u\left(t, x, \xi\right) + \frac{\partial z_b^\star}{\partial x} u\left(t, x, z_b^\star\right) + w\left(t, x, \xi\right) - w\left(t, x, z_b^\star\right) = \frac{\partial h}{\partial t} - \frac{dF_\xi}{dt} + \frac{dF_b}{dt}$$

The evolution of the water depth finally writes as

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(h\overline{u}\right) = \frac{dF_\xi}{dt} - \frac{dF_b}{dt} \tag{14}$$

In the context of hydrostatic approximation (long wave approximation), the pressure is given by $p = p_0 - \rho g \left(z - \xi\right)$. where $p_0$ is the atmospheric pressure at the free surface and the density $\rho$ constant in the vertical direction. Assuming that $p_0$ is constant in space, the gradient of the pressure is defined by

$$\frac{\partial p}{\partial x} = \rho g \frac{\partial \xi}{\partial x} + \left(\xi - z\right) g \frac{\partial \rho}{\partial x} \quad \text{and} \quad \int_{z_b^\star}^{\xi} \frac{1}{\rho}\frac{\partial p}{\partial x}dz = hg\frac{\partial \xi}{\partial x} + \frac{gh^2}{2\rho}\frac{\partial \rho}{\partial x}$$

Using the conservation of the mixture density and the divergence free assumption, the first equation of the momentum writes as

$$\frac{\partial u}{\partial t} + 2\frac{\partial K}{\partial x} + \frac{\partial}{\partial z}\left(uw\right) + \frac{1}{\rho}\frac{\partial p}{\partial x} = \boldsymbol{\mathcal{F}}_x \quad \text{where} \quad \boldsymbol{\mathcal{F}}_x = \frac{1}{\rho}\mathbf{F}_x + \frac{1}{\rho}\frac{\partial \tau_{xx}}{\partial x} + \frac{1}{\rho}\frac{\partial \tau_{xz}}{\partial z} \quad \text{and} \quad K = \frac{uu}{2}$$

Averaging this equation we obtain

$$\frac{\partial}{\partial t}\left(h\overline{u}\right) + \frac{\partial}{\partial x}\left(2h\overline{K}\right) + gh\frac{\partial \xi}{\partial x} + \frac{gh^2}{2\rho}\frac{\partial \rho}{\partial x} = h\overline{\boldsymbol{\mathcal{F}}}_x + \left(\frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial x}u - w\right)u\left(t, x, \xi\right)$$
$$- \left(\frac{\partial z_b^\star}{\partial t} + \frac{\partial z_b^\star}{\partial x}u - w\right)u\left(t, x, z_b^\star\right)$$

Then, still using the kinematic conditions, we derive that

$$\frac{\partial}{\partial t}\left(h\overline{u}\right) + \frac{\partial}{\partial x}\left(2h\overline{K}\right) + gh\frac{\partial \xi}{\partial x} + \frac{gh^2}{2\rho}\frac{\partial \rho}{\partial x} = h\overline{\boldsymbol{\mathcal{F}}}_x + \frac{dF_\xi}{dt}u\left(t, x, \xi\right) - \frac{dF_b}{dt}u\left(t, x, z_b^\star\right) \tag{15}$$

where

$$\overline{K} = \frac{\overline{uu}}{2} + \frac{P}{2} \quad \text{with} \quad P = \overline{u'u'}$$

We can see that the average $\overline{K}$ is not completely defined for us, as we still need to deal with non averaged components of the velocity. The classical shallow water model is obtained by assuming that $\overline{u'u'}$ is negligible. In order to take into account some amount of vertical shear, we will now derive an equation for the average $\overline{K}$. Starting from the momentum equation, we can derive the following set of equations,

$$\frac{\partial K}{\partial t} + u\frac{\partial}{\partial x}\left(uu\right) + u\frac{\partial}{\partial z}\left(uw\right) + \frac{u}{\rho}\frac{\partial p}{\partial x} = u\boldsymbol{\mathcal{F}}_x$$

Using the divergence free relation, this equation also writes as

$$\frac{\partial K}{\partial t} + \frac{\partial}{\partial x}\left(uK\right) + \frac{\partial}{\partial z}\left(wK\right) + gu\frac{\partial \xi}{\partial x} + \frac{\xi - z}{\rho}gu\frac{\partial \rho}{\partial x} = u\boldsymbol{\mathcal{F}}_x$$

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where the hydrostatic condition has been used. Averaging this equation gives

$$
\frac{\partial}{\partial t}\left(h\overline{\mathrm{K}}\right) + \frac{\partial}{\partial x}\left(h\overline{\mathrm{K}u}\right) + gh\overline{u}\frac{\partial \xi}{\partial x} + \frac{gh^2}{2\rho}\overline{u}\frac{\partial \rho}{\partial x} = -2g\frac{\partial \rho}{\partial x}\overline{(\eta - x_3)u'} + h\overline{u\boldsymbol{\mathcal{F}}_x} + \frac{d\mathrm{F}_\xi}{dt}\mathrm{K}\left(t, x, \xi\right)
$$
$$
- \tfrac{d\mathrm{F}_b}{dt}\mathrm{K}\left(t, x, \mathrm{z}_b^\star\right)
$$

where

$$
2\overline{\mathrm{K}u} = 2\overline{\mathrm{K}}\,\overline{u} + 2\mathrm{P}\overline{u} + \overline{u'u'u'}
$$

The third order fluctuations is here assumed to be smaller than the second order fluctuations such that it allows us to formulate the third order fluctuations as a gradient, to produce dissipation of the depth average energy [**Gavrilyuk_et_Al_2018**, **Praveen_et_Al_2020** ] :

$$
\overline{u'u'u'} \simeq -2\kappa\frac{\partial \mathrm{P}}{\partial x} \qquad \Longrightarrow \qquad \overline{\mathrm{K}u} = \overline{\mathrm{K}}\,\overline{u} + \mathrm{P}\overline{u} - \kappa\frac{\partial \mathrm{P}}{\partial x} \tag{16}
$$

Moreover, we assume that $\overline{u\boldsymbol{\mathcal{F}}_x} \simeq \overline{u}\mathrm{F}$ and the evolution of the average energy writes as

$$
\frac{\partial}{\partial t}\left(h\overline{\mathrm{K}}\right) + \frac{\partial}{\partial x}\left(h\overline{u}\left(\overline{\mathrm{K}} + \mathrm{P}\right)\right) + gh\overline{u}\frac{\partial \xi}{\partial x} + \frac{gh^2}{2\rho}\overline{u}\frac{\partial \rho}{\partial x} = -2g\frac{\partial \rho}{\partial x}\overline{(\eta - x_3)u'} + h\overline{u}\mathrm{F} + \frac{\partial}{\partial x}\left(\kappa\frac{\partial \mathrm{P}}{\partial x}\right)
$$
$$
+ \quad \frac{d\mathrm{F}_\xi}{dt}\mathrm{K}_\xi - \frac{d\mathrm{F}_b}{dt}\mathrm{K}_{\mathrm{z}_b^\star} \tag{17}
$$

We can now use the relation (16) reformulate the bedload interface evolution (8) as

$$
\frac{\partial \mathrm{z}_b^\star}{\partial t} + \frac{2\mu}{1 - \phi_s}\frac{\partial}{\partial x}\left(\overline{\mathrm{K}}\,\overline{u} + \mathrm{P}\overline{u}\right) = \frac{\mathcal{D} - \mathcal{E}}{1 - \phi_s} + \frac{2\mu}{1 - \phi_s}\frac{\partial}{\partial x}\left(\kappa\frac{\partial \mathrm{P}}{\partial x}\right) \tag{18}
$$

There is an alternative formulation, using the relation (10), that writes as

$$
\frac{\partial \mathrm{z}_b^\star}{\partial t} + u\left(t, x, \mathrm{z}_b^\star\right)\frac{\partial \mathrm{z}_b^\star}{\partial x} = \frac{\mathcal{D} - \mathcal{E}}{1 - \phi_s} + \frac{2\mu}{1 - \phi_s}\frac{\partial}{\partial x}\left(\kappa\frac{\partial \mathrm{P}}{\partial x}\right) + w\left(t, x, \mathrm{z}_b^\star\right) \tag{19}
$$

In this context, $(u\left(t, x, \mathrm{z}_b^\star\right))$ and vertical $(w\left(t, x, \mathrm{z}_b^\star\right))$ velocities at the bedload should be defined as functions of the averaged quantities characterizing the fluid just above.

The equation for volume concentration, when averaged, gives

$$
\frac{\partial h\overline{\alpha_s}}{\partial t} + \frac{\partial}{\partial x}\left(h\overline{u\alpha_s}\right) = \left(\frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial x}u - w\right)\alpha_s\left(t, x, \xi\right) - \left(\frac{\partial \mathrm{z}_b^\star}{\partial t} + \frac{\partial \mathrm{z}_b^\star}{\partial x}u - w\right)\alpha_s\left(t, x, \mathrm{z}_b^\star\right)
$$

we assume that $\alpha_s\left(t, x, \xi\right) = 0$ and $\alpha_s\left(t, x, \mathrm{z}_b^\star\right) = 1 - \phi_s$. To compute the average $\overline{u\alpha_s}$, we use a Fick's law approximation written as

$$
\overline{u\alpha_s} \simeq \overline{u}\,\overline{\alpha_s} - \gamma\frac{\partial \overline{\alpha_s}}{\partial x}
$$

where $\gamma$ is a positive coefficient. The final averaged equation writes as

$$
\frac{\partial}{\partial t}\left(h\overline{\alpha_s}\right) + \frac{\partial}{\partial x}\left(h\overline{\alpha_s}\,\overline{u}\right) = \frac{\partial}{\partial x}\left(\gamma\frac{\partial \overline{\alpha_s}}{\partial x}\right) - \left(\mathcal{D} - \mathcal{E}\right)
$$

All together the depth-averaged model for sediment flows writes as

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(h\overline{u}) = \frac{d\mathrm{F}_\xi}{dt} - \frac{d\mathrm{F}_b}{dt} \tag{20}$$

$$\frac{\partial}{\partial t}(h\overline{u}) + \frac{\partial}{\partial x}(2h\mathrm{K}) + gh\frac{\partial\xi}{\partial x} + \frac{gh^2}{2\rho}\frac{\partial\rho}{\partial x} = h\mathrm{F} + \frac{d\mathrm{F}_\xi}{dt}u_\xi - \frac{d\mathrm{F}_b}{dt}u_{\mathrm{z}_b^\star} \tag{21}$$

$$\frac{\partial}{\partial t}(h\overline{\mathrm{K}}) + \frac{\partial}{\partial x}(h\overline{u}(\overline{\mathrm{K}}+\mathrm{P})) + gh\overline{u}\frac{\partial\xi}{\partial x} + \frac{gh^2}{2\rho}\overline{u}\frac{\partial\rho}{\partial x} = -2g\frac{\partial\rho}{\partial x}\overline{(\eta-x_3)u'} + h\overline{u}\mathrm{F} + \frac{d\mathrm{F}_\xi}{dt}\mathrm{K}_\xi - \frac{d\mathrm{F}_b}{dt}\mathrm{K}_{\mathrm{z}_b^\star}$$
$$+ \frac{\partial}{\partial x}\left(\kappa\frac{\partial\mathrm{P}}{\partial x}\right) \tag{22}$$

$$\frac{\partial}{\partial t}(h\overline{\alpha_s}) + \frac{\partial}{\partial x}(h\overline{u}\,\overline{\alpha_s}) = -\phi_f\frac{d\mathrm{F}_b}{dt} + \frac{\partial}{\partial x}\left(\gamma\frac{\partial\overline{\alpha_s}}{\partial x}\right) \tag{23}$$

$$\frac{\partial\mathrm{z}_b^\star}{\partial t} + \frac{2\mu}{\phi_f}\frac{\partial}{\partial x}(\overline{\mathrm{K}}\overline{u}+\mathrm{P}\overline{u}) = \frac{d\mathrm{F}_b}{dt} + \frac{2\mu}{\phi_f}\frac{\partial}{\partial x}\left(\kappa\frac{\partial\mathrm{P}}{\partial x}\right) \tag{24}$$

where $\phi_f + \phi_s = 1$.

$$\xi = \mathrm{z}_b^\star + h, \quad \rho = \overline{\alpha_s}\rho_s + (1-\overline{\alpha_s})\rho_f = \rho_f + \delta\rho\overline{\alpha_s} \quad \text{and} \quad \delta\rho = \rho_s - \rho_f$$

Note that, in the case where we constraint $\mathrm{P} = 0$, when vertical fluctuations of the velocity are neglected, the equation for the energy $\overline{\mathrm{K}}$ is useless and we recover the model used in [**Liu_et_Al_2015** ]. If in addition we take $\mu = 0$, then the present model degenerates to the one used in [**Simpson_and_Castelltort_2006** ].
Here we can for instance take $u(t, x, \mathrm{z}_b^\star)$ as:

$$u(t, x, \mathrm{z}_b^\star) = u_s\frac{1}{1-Fr^2}\frac{\overline{u}}{|\overline{u}|}, \tag{25}$$

where $u_s$ is the velocity of sediment, $Fr$ is the Froudre number (compute for shear shallow water). The characteristic velocity of advection of body sedimentary given by (25) allows to take into account a phase lag between sediment velocity and water velocity.
Therefore, the proposed model is a extensions of those modeling The model we propose here is an extension of these reduced models, taking into account the effects of vertical fluctuations of the horizontal velocity. This model can also be put in the following compact form

$$\frac{\partial\mathbf{W}}{\partial t} + \frac{\partial\mathbf{F}}{\partial x} + gh\mathbf{B}_x^\mathrm{K}\frac{\partial h}{\partial x} + gh\mathbf{B}_x^\star\frac{\partial\mathrm{z}_b^\star}{\partial x} + \frac{gh^2\delta\rho}{2\rho}\mathbf{B}_x^\star\frac{\partial\overline{\alpha_s}}{\partial x} = \mathbf{S} \tag{26}$$

where

$$\mathbf{W} = \begin{pmatrix} h \\ h\overline{u} \\ h\overline{\mathrm{K}} \\ h\overline{\alpha_s} \\ \mathrm{z}_b^\star \end{pmatrix}, \qquad \mathbf{F} = \begin{pmatrix} h\overline{u} \\ 2h\overline{\mathrm{K}} + \frac{gh^2}{2} \\ h\overline{u}(\overline{\mathrm{K}}+\mathrm{P}) \\ h\overline{u}\,\overline{\alpha_s} \\ \frac{2\mu(\overline{\mathrm{K}}\overline{u}+\mathrm{P}\overline{u})}{\phi_f} \end{pmatrix}, \qquad \mathbf{B}_x^\mathrm{K} = \begin{pmatrix} 0 \\ 0 \\ \overline{u} \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{B}_x^\star = \begin{pmatrix} 0 \\ 1 \\ \overline{u} \\ 0 \\ 0 \end{pmatrix}$$

and **S** is the vector of the remaining contributions including the effects of erosion/deposition, rainfall/evaporation, external forces and dissipations. The system (26) is genuinely non conservative and its numerical approximation, in the context of finite volumes schemes, needs some specific treatments [**Praveen_et_Al_2020** ].The Jacobian matrix associated to the conservative

flux $\mathbf{F}$ is diagonalizable (Hyperbolic system). When adding the non conservative contribution associated to the derivative of $z_b^\star$ the hyperbolicity of the left hand size is no more guaranty. This will lead to further numerical complications. The system (26) was obtained using the formulation (18). When using the alternative formulation for the bedload dynamic (19), we get the following system

$$\frac{\partial \mathbf{W}}{\partial t} + \frac{\partial \widetilde{\mathbf{F}}}{\partial x} + gh\mathbf{B}_x^{\text{K}}\frac{\partial h}{\partial x} + \widetilde{\mathbf{B}}_x^{\star}\frac{\partial z_b^\star}{\partial x} + \frac{gh^2\delta\rho}{2\rho}\mathbf{B}_x^\star\frac{\partial \overline{\alpha_s}}{\partial x} = \mathbf{S} \tag{27}$$

where

$$\mathbf{W} = \begin{pmatrix} h \\ h\overline{u} \\ h\overline{\text{K}} \\ h\overline{\alpha_s} \\ z_b^\star \end{pmatrix}, \qquad \widetilde{\mathbf{F}} = \begin{pmatrix} h\overline{u} \\ 2h\overline{\text{K}} + \frac{gh^2}{2} \\ h\overline{u}\left(\overline{\text{K}} + \text{P}\right) \\ h\overline{u}\,\overline{\alpha_s} \\ 0 \end{pmatrix}, \qquad \widetilde{\mathbf{B}}_x^\star = \begin{pmatrix} 0 \\ gh \\ gh\overline{u} \\ 0 \\ u_{\text{b}}^\star \end{pmatrix}$$

with $u_{\text{b}}^\star = u\left(t, x, z_b^\star\right)$. The terms on the left side of the equation (27) define a hyperbolic non-conservative system whose eigenvalues are: $u$, $u_{\text{b}}^\star$, $u - \sqrt{gh + 3\text{P}}$ and $u + \sqrt{gh + 3\text{P}}$. We recover the eigenvalues obtain in [**Gavrilyuk_et_Al_2018** ] and in addition two other waves associated to sediment concentration and to the bedload dynamic.

## III CONCLUSION

We have performed here the mathematical derivation of a physical model for sediment transport in shear shallow flow. This model, inspired by very recent works, is an extension of the classical sediment transport models and takes into account the effects of horizontal velocity shear. As a result, the validity regime of the present model is extended, which gives a more appropriate framework for the study of coastal flows. The model described here assumes quasi 2D flows and after averaging gives 1D equations. There are no particular difficulties to extend this approach to 3D flows. This last step does not pose any particular difficulties and will be necessary for concrete applications. The proposed models will then be the subject of numerical approximations in the context of finite volume, then numerical simulations associated with the target watershed of Douala.

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