An intrinsic PGD for parametric elliptic problems

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Sketch

1. PGD for parametric symmetric elliptic problems
   - Existence of PGD solution
   - A deflation algorithm to compute the solution.
   - Numerical tests.
2. Concluding remarks.

Best Data tensorisation

Let V a Hilbert space, $\mathcal{A}$ a bilinear form on V and $\mathcal{L}$ a linear form on V. Let

$$\forall u, v \in V, \quad \mathcal{A}(u,v) = \mathcal{L}(v) \tag{1}$$

Let $\mathcal{S} = \{z(x_1, ..., x_n) = \prod_{i=1}^{n} X_i(x_i)\}$ and $\mathcal{S}_m = \{v = \sum_{j=1}^{m} z_j, z_i \in \mathcal{S}\}$.

Problem

$\Rightarrow$ Find the best approximation $u_m \in \mathcal{S}_m$ of u solution of eq. (1).

- Additional constraint : orthogonality (otherwise ill-posed for $d \geq 3$).
- Normalise all modes w_i^j but one.

Post-processing : data reduction

Let $\mathcal{A}(u,v) = \int uv$ and $\mathcal{L}(v) = \int fv$. The problem reads :

$$u_m \in \arg \min_{u^* \in \mathcal{S}_m} \|f - u^*\|_2 \tag{2}$$

Parametric elliptic problems

Let

- A separable Hilbert space $(H, (\cdot,\cdot))$.
- A measure space $(\Gamma, \mathcal{B}, \mu)$, with standard notations, so that μ is σ-finite.
- A form $a \in L^\infty(\Gamma, B_s(H); d\mu)$ uniformly elliptic and coercive on H.
- A data function $f \in L^2(\Gamma, H'; d\mu)$

We are interested in solving the variational problem:

Find $u(\gamma) \in H$ such that

$$a(u(\gamma), v; \gamma) = \langle f(\gamma), v\rangle_{H'-H}, \quad \forall v \in H, \ d\mu\text{-a.e. } \gamma \in \Gamma, \qquad (3)$$

This is a common situation in many engineering problems when the properties of the media are parameter-depending.

M. Azaïez et al, A new Algorithm of proper generalized decomposition for parametrics symmetric elliptic problems. SIAM J. MATH. ANAL. Vol. 50, No. 5, pp. 5426–5445 (2018)

Examples

Case 1 : $H = H_0^1(\Omega)$ and $\gamma \in [0, 1]$

$$-\Delta u + \gamma u = f, \text{ on } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

Case 2: $H = H_0^1(\Omega)$ and $\gamma \in [0.01, 1]$

$$-\nabla \cdot (\nu(x, \gamma) \nabla u) = f, \text{ on } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad \Omega = [0, 1]^2$$

with

$$\nu(x, \gamma) = \begin{cases} 1 & \text{if} \quad 0 \leq x \leq 1/2 \\ \gamma & \text{if} \quad 1/2 < x \leq 1 \end{cases}$$

Parametric elliptic problems

Considering the problem :

Find $\tilde{u} \in L^2(\Gamma, H; d\mu)$,

$$\bar{a}(\tilde{u}, v) = \int_\Gamma \langle f(\gamma), v(\gamma) \rangle \, d\mu(\gamma), \ \forall \, v \in L^2(\Gamma, H; d\mu), \tag{4}$$

where

$$\bar{a}(v, w) = \int_\Gamma a(v(\gamma), w(\gamma); \gamma) \, d\mu(\gamma), \quad \forall v, \ w \in L^2(\Gamma, H; d\mu) \tag{5}$$

It follows $\tilde{u} = u \quad d\mu$-a.e. $\gamma \in \Gamma$, and then $u \in L^2(\Gamma, H; d\mu)$

Proper Generalized Decomposition

The PGD searches a tensorized decomposition to Approximate

$$u(\gamma) \simeq \sum_{k\geq 1} \Phi_k(\gamma)\, w_k,\;\; w_k \in H.$$

PGD computes the w_k online and iteratively.

- This is done by partial Galerkin problems.
- For instance, the first summand $u_1(\gamma) = \Phi_1(\gamma)\, w_1$, with $\Phi_1(\gamma) \in L^2(\Gamma, d\mu)$ and $w_1 \in H$ is a solution of

$$\begin{aligned} a(\Phi_1(\gamma)\, w_1, \Phi_1(\gamma)\, v) &= \langle f(\gamma), \Phi_1(\gamma)\, v\rangle,\; \forall v \in H, d\mu\text{-a.e. } \gamma \in \Gamma; \\ \int_\Gamma a(\Phi_1(\gamma)\, w_1, w_1)\, s(\gamma) d\mu(\gamma) &= \int_\Gamma \langle f(\gamma), w_1\rangle\, s(\gamma) d\mu(\gamma),\; \forall s \in L^2(\Gamma, d\mu). \end{aligned}$$

- These problems are (can be) solved by a power-iteration algorithm

Proper Generalized Decomposition

The following term

$$u_k(\gamma) = \sum_{j=1}^{k} \Phi_j(\gamma)\, w_j, = u_{k-1} + \Phi_k(\gamma)\, w_k \;\; w_k \in H.$$

is computed by a deflation algorithm, i. e., the same but replacing f by the current residual, $r_{k-1}(\gamma) = f(\gamma) - A(\gamma)u_{k-1}$.

- The PGD has been characterized as a descent method for elliptic problems (Falcó and Nouy, 2016).

A. Falcó and A. Nouy, *Proper generalized decomposition for nonlinear convex problems in tensor Banach spaces*. Numer. Math. **121** (2012), 503–530.

Optimal sub-spaces of finite dimension

Problem targeted: Find the best subspace W of H of dimension $\leq k$ that minimizes the mean quadratic error between $u(\gamma)$ and $u_W(\gamma)$ with respect to the norm generated by the form $a(\cdot,\cdot;\gamma)$.
That is, W solves

$$\text{(P)} \qquad \min_{Z \in \mathcal{S}_k} \int_\Gamma a(u(\gamma) - u_Z(\gamma), u(\gamma) - u_Z(\gamma); \gamma)\, d\mu(\gamma), \tag{6}$$

where $\mathcal{S}_k$ is the family of subspaces of H of dimension $\leq k$ and $u_Z(\gamma) \in Z$ is Galerkin projection of u on Z,

$$a(u_Z(\gamma), z; \gamma) = \langle f(\gamma), z \rangle, \quad \forall z \in Z,\ d\mu\text{-a.e. } \gamma \in \Gamma.$$

Some properties -due to the symmetry of $a(\cdot,\cdot)$-

- For every closed subspace $Z \subset H$, the function u_Z defined as

$$u_Z \in L^2(\Gamma, H; d\mu), \quad \bar{a}(u_Z, z) = \int_\Gamma < (f(\gamma), z(\gamma))\, d\mu(\gamma), \ \forall z \in L^2(\Gamma, H; d\mu)$$

is the unique solution of

$$\min_{z \in L^2(\Gamma, Z; d\mu)} \bar{a}(u - z, u - z).$$

- Moreover, for $d\mu$-a.e. $\gamma \in \Gamma$, the vector $u_Z(\gamma)$ is the solution of

$$\min_{w \in Z} a(u(\gamma) - w, u(\gamma) - w; \gamma).$$

- The subspace $W \in \mathcal{S}_k$ solves problem (P) if and only if it is a solution of

$$\max_{Z \in \mathcal{S}_k} \int_\Gamma \langle f(\gamma), u_Z(\gamma) \rangle \, d\mu(\gamma).$$

A look at the 1D case

When $k = 1$, problem (P) can be written as

$$\min_{v\in H, \varphi\in L^2(\Gamma;d\mu)} \int_\Gamma a(u(\gamma)-\varphi(\gamma)v, u(\gamma)-\varphi(\gamma)v;\gamma)d\mu(\gamma).$$

So, taking the derivative of the functional

$$(v,\varphi)\in H\times L^2(\Gamma;d\mu)\mapsto \int_\Gamma a(u(\gamma)-\varphi(\gamma)v, u(\gamma)-\varphi(\gamma)v;\gamma)d\mu(\gamma),$$

we deduce that $W = Span(w)$, where w is a solution of the non-linear eigen-space problem

$$\int_\Gamma \frac{a(u(\gamma),w;\gamma)}{a(w,w,\gamma)} a(u(\gamma),v;\gamma)d\mu(\gamma) = \int_\Gamma \frac{a(u(\gamma),w;\gamma)^2}{a(w,w,\gamma)^2} a(w,v;\gamma)d\mu(\gamma), \tag{7}$$

$\forall\, v\in H.$

A look at the 1D case

- If a does not depend on γ, statement (7) is equivalent to

$$\mathcal{R}w = \int_\Gamma a(u(\gamma), w) u(\gamma) d\mu(\gamma) = \lambda\, w,$$

where

$$\lambda = \frac{\displaystyle\int_\Gamma a(u(\gamma), w)^2 d\mu(\gamma)}{a(w, w)}.$$

i.e. w is an eigenvector of the POD operator $\mathcal{R}$ when the inner product in H is the form $a(\cdot, \cdot)$.

- However, when a depends on γ problem (7) does not correspond to a proper eigenvalue equation: It is a non-linear eigenfunction problem, where no eigenvalues appear.

Then, we are considering a genuine extension of the POD.

Optimal sub-spaces

Theorem

There exists at least a sub-space $Z \in S_k$ that solves problem (P)

• Proof by direct method of the Calculus of Variations + compactness argument.

• Special proof for the 1D case. This problem is equivalent to

$$\text{(P')} \qquad \max_{\substack{\Psi \in H \\ \|\Psi\|=1}} \int_\Gamma \frac{\langle f(\gamma), \Psi \rangle^2}{a(\Psi, \Psi; \gamma)} \, d\mu(\gamma).$$

Theorem

Problem (P') admits at least one solution.

• Proof by combination of direct method of the Calculus of Variations, compactness in Hilbert spaces, uniform boundedness and ellipticity of forms $a(\cdot, \cdot; \gamma)$ and Fatou's Lemma.

Tensor approximation

Similarly to the PGD, we expand $u(\gamma)$ by the tensor approximation

$$u(\gamma) = \sum_{k\geq 1} \Phi_k(\gamma)\, w_k,\;\; w_k \in H.$$

where the w_k are obtained by a deflation algorithm similar to the one followed by PGD:

- Initialization:

$$w_1 = \mathrm{argmin}_{\Psi\in H} \int_\Gamma a(u(\gamma) - u_\Psi(\gamma), u(\gamma) - u_\Psi(\gamma); \gamma)\, d\mu(\gamma),$$

  where u_Ψ is the Galerkin solution of the targeted elliptic problem on $span\{\Psi\}$.

- Iteration: Known $u_{k-1}(\gamma) = \sum_{i=1}^{k-1} \Phi_i(\gamma)\, w_i,$, let $e_{k-1} = u - u_{k-1}$.

$$w_k = \mathrm{argmin}_{\Psi\in H} \int_\Gamma a(e_{k-1}(\gamma) - u_\Psi(\gamma), e_{k-1}(\gamma) - u_\Psi(\gamma); \gamma)\, d\mu(\gamma),$$

Tensor approximation

- It holds that $w_k = (e_{k-1})_W$, with W a solution of

$$\max_{\Psi \in H} \int_\Gamma \langle f_{k-1}(\gamma), (e_{k-1})_\Psi(\gamma) \rangle \, d\mu(\gamma),$$

where $\langle f_{k-1}(\gamma), v \rangle = \langle f(\gamma), v \rangle - a(u_{k-1}(\gamma), v)$ and $(e_{k-1})_\Psi \in L^2(\Gamma, Z_\Psi; d\mu)$ is the solution of

$$\bar{a}((e_{k-1})_\Psi, z) = \int_\Gamma \langle f_{k-1}(\gamma), z(\gamma) \rangle \, d\mu(\gamma), \quad \forall z \in L^2(\Gamma, Z_\Psi; d\mu).$$

- This allows us to carry out the different iterations without needing to know the function u.

Theorem

The approximations u_n provided by the deflation algorithm strongly converge to the solution u of problem (P).

- Proof by orthogonality properties of residuals, consequence of the symmetry of forms a.

Solution of non-linear eigen-spaces problem (1D case)

- We look for the solution of

$$\int_\Gamma \frac{a(u(\gamma), w; \gamma)}{a(w, w, \gamma)} a(u(\gamma), v; \gamma) d\mu(\gamma) = \int_\Gamma \frac{a(u(\gamma), w; \gamma)^2}{a(w, w, \gamma)^2} a(w, v; \gamma) d\mu(\gamma),$$

$\forall\, v \in H,$
by the Power-Iteration method,

$$\int_\Gamma [\varphi(w^n, \gamma)]^2 a(w^{n+1}, v; \gamma) d\mu(\gamma) = \int_\Gamma \varphi(w^n, \gamma) a(u(\gamma), v; \gamma) d\mu(\gamma), \forall v \in H,$$

with $\varphi(w, \gamma) = \dfrac{a(u(\gamma), w; \gamma)}{a(w, w, \gamma)} = \dfrac{\langle f(\gamma), w \rangle}{a(w, w, \gamma)}$.

- One may prove that

$$\varphi(w^n, \gamma) \neq 0 \text{ a. e. } d\mu(\gamma) \implies \varphi(w^{n+1}, \gamma) \neq 0 \text{ a. e. } d\mu(\gamma).$$

Then the iteration method is well defined.

Solution of non-linear eigen-spaces problem

- We are finding an eigen-space of the operator $T : H \mapsto H$ given by

$$\int_\Gamma [\varphi(w,\gamma)]^2 a(T(w), v; \gamma) d\mu(\gamma) = \int_\Gamma \varphi(w,\gamma) a(u(\gamma), v; \gamma) d\mu(\gamma), \forall v \in H.$$

Note that $T(\lambda w) = \lambda T(w)$ as $\varphi(\lambda w, \gamma) = \lambda^{-1} \varphi(w,\gamma)$.

Numerical tests

We have tested the Algorithm for 2D academic elliptic problems:

Test 1

$$-\Delta u + \gamma u = f, \text{ on } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

with $\gamma \in [0, 1]$, $f = 1$.

$\rightarrow$ The integrals on Γ have been approximated by the mid-point rule.

- The PI Method has been used to solve the eigen-space problem.
- Results:
  - The Power Iteration Method converges with linear rate ($\simeq 1/4$).
  - The series $\sum_{i \geq 1} \Phi_i(\gamma)\, w_i(x)$ converges to the solution $u(x, \gamma)$ with spectral rate (the error is roughly divided by 10^4 at each iteration).

Numerical tests

Test 2

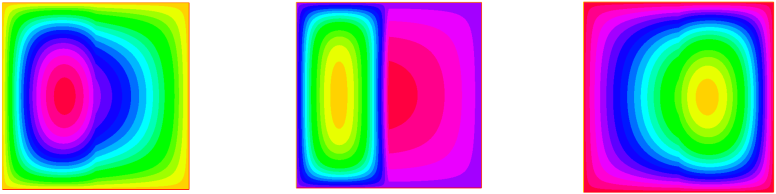
$$-\nabla \cdot (\nu(x,\gamma)\,\nabla u) = f, \text{ on } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad \Omega = [0,1]^2$$

with

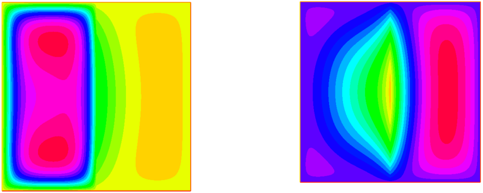
$$\nu(x,\gamma) = \begin{cases} 1 & \text{if} \quad 0 \le x \le 1/2 \\ \gamma & \text{if} \quad 1/2 < x \le 1 \end{cases} \quad \text{with } \Gamma = [0.01, 1], \ f = 1.$$

- Results:
  - The Power Iteration Method converges with linear rate ($\simeq 1/2$).
  - The series $\sum_{i\ge 1} \Phi_i(\gamma)\, w_i(x)$ converges to the solution $u(x,\gamma)$ with spectral rate (the error is roughly divided by 430 at each iteration).

Eigenfunctions



Eigenfunctions w_1, w_2, w_3.



Eigenfunctions w_4, w_5.

Computation of PGD modes

The PGD method builds an iterative approximation of the solution of (4) by a deflation approach. It approximates the solution u by a series

$$u(\gamma) = \sum_{i\geq 1} \varphi_i(\gamma)\, w_i,$$

where the pairs $(\varphi_i, w_i) \in L^2(\Gamma, d\mu) \times H$ are recursively obtained as a solution of the non-linear coupled problems

$$\left\{ \begin{array}{lcll} \bar{a}(\varphi_i\, w_i, \varphi_i\, v) & = & \langle \bar{f}_i, \varphi_i\, v\rangle & \forall v \in H, \\ \bar{a}(\varphi_i\, w_i, \psi\, w_i) & = & \langle \bar{f}_i, \psi\, w_i\rangle & \forall \psi \in L^2(\Gamma, d\mu). \end{array} \right. \tag{8}$$

For all $i = 1, 2, \cdots$, all problems (8) fit within the following abstract formulation: Find $(\varphi, w) \in L^2(\Gamma, d\mu) \times H$ solution of

$$\left\{ \begin{array}{lcll} \bar{a}(\varphi\, w, \varphi\, v) & = & \langle \bar{f}, \varphi\, v\rangle & \forall v \in H, \\ \bar{a}(\varphi\, w, \psi\, w) & = & \langle \bar{f}, \psi\, w\rangle & \forall \psi \in L^2(\Gamma, d\mu). \end{array} \right. \tag{9}$$

Computation of PGD modes

To solve (9) we will apply a PI algorithm with normalization. Consider $v \in H$ and denote by $\varphi(v) = \varphi(v, \cdot) \in L^2(\Gamma, d\mu)$ and $z = T(v) \in H$ the solutions of the problems

$$\bar{a}(\varphi(v)\, v, \psi\, v) = \langle \bar{f}, \psi\, v\rangle \quad \forall \psi \in L^2(\Gamma, d\mu); \tag{10}$$

$$\bar{a}(\varphi(v)\, z, \varphi(v)\, u) = \langle \bar{f}, \varphi(v)\, u\rangle \quad \forall u \in H. \tag{11}$$

PI Algorithm with normalization is stated as follows:
Initialization: Give a non-zero $w^0 \in H$ such that $\varphi^0 = \varphi(w^0)$ is non-zero in $L^2(\Gamma, d\mu)$.

Iteration: Known a non-zero $w^n \in H$ such that $\varphi^n = \varphi(w^n)$ is non-zero in $L^2(\Gamma, d\mu)$, compute

$$\begin{aligned} &\text{a)} \qquad \widetilde{w}^{n+1} = T(w^n) \in H \text{ and } w^{n+1} = \frac{\widetilde{w}^{n+1}}{\|\widetilde{w}^{n+1}\|} \in H; \\ &\text{b)} \qquad \varphi^{n+1} = \varphi(w^{n+1}) \in L^2(\Gamma, d\mu). \end{aligned} \tag{12}$$

Here $\varphi \in L^2(\Gamma, d\mu)$ "is not zero" means is not zero $d\mu$ a. e. $\gamma \in \Gamma$.

Error estimates for the PI Algorithm

Let us introduce, for $\varepsilon > 0$, the sets

$$\mathcal{E}_\varepsilon = \{\psi \in L^2(\Gamma; d\mu) \text{ s. t. } \|\psi\|_{L^2(\Gamma;d\mu)} \geq \varepsilon\},$$

Define the functions

$$\begin{aligned}
\Phi_1(\xi) &= \left(1 + \frac{M}{\alpha}\,\frac{2+\xi}{(1-\xi)^2}\right), \; \Phi_2(\xi) = \frac{M}{\alpha}\left(1 + \frac{1}{1-\xi}\right)\Phi_1(\xi),\\
\sigma(\xi,\zeta) &= \left(\Phi_1(\xi) + \frac{\Phi_2(\xi)}{1-\xi}\left(\frac{\zeta}{\alpha\,\varepsilon}\right)^2\right)\left(\frac{\zeta}{\alpha\,\varepsilon}\right)^2,\\
\delta(\xi,\zeta) &= \frac{1}{2}\left(\frac{1}{1-\sigma(\xi,\zeta)\,\xi}\,\Phi_1(\xi) + \Phi_2(\xi)\right)\left(\frac{\zeta}{\alpha\,\varepsilon}\right)^2;
\end{aligned}$$

where α and M are the constants of ellipticity and continuity.
Note that, if $\xi \neq 1$

$$\lim_{\zeta\to 0}\sigma(\xi,\zeta) = \lim_{\zeta\to 0}\delta(\xi,\zeta) = 0.$$

Error estimates for the PI Algorithm

Theorem

Assume that $f \neq 0$. Assume that the sequence provided by the PI Algorithm satisfies $\varphi^n \in \mathcal{E}_\varepsilon$, for some $\varepsilon > 0$. Assume that

$$\sigma(r, \|f\|_{L^2(\Gamma,H';d\mu)}) < r^{-1} \quad \text{and} \quad \delta(r, \|f\|_{L^2(\Gamma,H';d\mu)}) < 1$$

for some $r \in (0,1)$. Then if $z \in B_H(w,r)$ is such that $\varphi(z) \neq 0$, it holds $\dfrac{T(z)}{\|T(z)\|} \in B_H(w,r)$ and

$$\left\| \frac{T(z)}{\|T(z)\|} - w \right\| \leq \Delta \, \|z - w\|. \tag{13}$$

where $\Delta = \delta(r, \|f\|_{L^2(\Gamma,H;d\mu)})$. Consequently the Power Iteration Algorithm to solve problem (9) converges with linear rate to w, the error estimate

$$\|w^n - w\| \leq \Delta^n \, \|w^0 - w\| \quad \text{if } w^0 \in B_H(w,r) \tag{14}$$

Error estimates for the PI Algorithm

Remark

Note that from Theorem 4, it holds

- *The convergence rate of the PI Algorithm is bounded as*

$$\Delta \leq C_1\, \rho^{-2} + o(\rho^{-2}),$$

  where $\rho = \dfrac{\|f\|_{L^2(\Gamma, H'; d\mu)}}{\alpha\, \varepsilon}$, *and* $\displaystyle\lim_{\tau \to \infty} \frac{o(\tau)}{\tau} = 0$, *for some constant* C_1.
- *If* f *is changed into* cf *for some constant* c, *then the functions* φ^n *are changed into* $c\varphi^n$. *Consequently,* ε *is changed into* $|c|\varepsilon$, *and therefore the bound* Δ *for the convergence rate does not change under re-scaling of the r. h. s.* f.
- *The Theorem 4 also holds when the form* a *is non-symmetric, its symmetry is assumed nowhere in its proof.*

Numerical experiments

We consider the solution of

$$\begin{cases} -\nabla \cdot (\mu(\gamma)\, \nabla u) & = & f & \text{in} & \Omega, \\ u & = & 0 & \text{on} & \partial\Omega; \end{cases} \tag{15}$$

where

$$\Omega = (0,1)^2 \text{ and } \quad \mu(\gamma)(\mathbf{x}) = \begin{cases} \gamma + \alpha_{min} & \text{if} & 0 \le x \le 1/4, \\ 1 + \alpha_{min} & \text{if} & 1/4 \le x \le 1. \end{cases} \quad \text{for all } \mathbf{x} \in \overline{\Omega}. \tag{16}$$

α_{min} is a real number to be selected to vary the minimum of $\mu(\gamma)$ (it is right the α in Theorem 4).
In sequel $H = H_0^1(\Omega)$ we have set $\Gamma = [0.01, 1]$ and

$$\gamma_i = 0.01 + \frac{0.99}{N}\,(i - \frac{1}{2}), \quad i = 1, \cdots, N.$$

Numerical experiments : convergence behaviour of the PI

We fix α_{min} such that $\alpha = 1$. As expected the PI Algorithm, converges with linear rate. Table 1 displays the numerical convergence rates, estimated by

$$r = \frac{\|w^{n+1} - w^n\|_H}{\|w^n - w^{n-1}\|_H}. \tag{17}$$

Iteration	Mode 1	Mode 2	Mode 3
1	75,30	104,44	24,31
2	73,57	22,13	12,04
3	–	22,1	22,09
4	–	–	22,09

Table: Convergence rates of PI Algorithm (12) to compute the three first PGD modes for problem (15).

M. Azaïez, T. Chacón Rebollo and M. Gómez Mármol, *On the computation of PGD modes of parametric elliptic problems*. SEMA, 2019

Numerical experiments : convergence of the PGD series

In this test we study the convergence of the truncated series u_i to the parametric solution $u(\gamma)$. The errors are measured in $L^2(\Gamma, L^2(\Omega), d\mu)$ and $L^2(\Gamma, H^1_0(\Omega), d\mu)$ norms. We observe a spectral convergence, possibly consequence of the analyticity of $u(\gamma)$ as a function of γ with values in H.

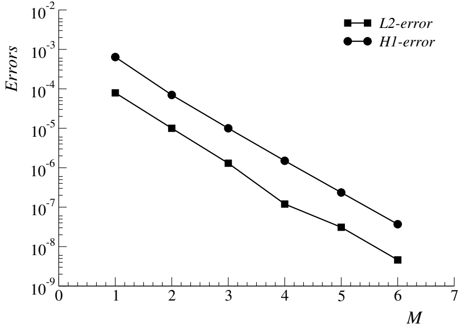


Figure: Convergence history of the modes u_i to $u(\gamma)$

Numerical experiments : effect of the viscosity

This experiment studies the dependence of the convergence rate on the minimum of the viscosity. We fix the values of α_{min} in order to have $\alpha = 1$, 2, and 4, 16, 32, 1000 and 2000. Table 2 displays the results for the two first modes, also displaying the ratio of convergence rates between consecutive values of α.

α	Mode 1	ratio	Mode 2	ratio
1	75	-	22	-
2	200	2.67	68	3.09
4	630	3.15	230	3.38
8	2200	3.49	854	3.66
16	8200	3.72	3250	3.80
32	31340	3.82	12680	3.90
100	302759	-	-	-
200	1204188	3.98	-	-

Table: Convergence rates of PI Algorithm to compute the 2 first PGD modes vs α. The ratios are the quotient between two consecutive convergence rates.

Numerical experiments : non-symmetric case

We study the convergence rate of the PI algorithm to compute the PGD modes for a non-symmetric parametric elliptic problem. We consider the advection-diffusion equations where the Péclet number acts as the parameter γ in our theory. This problem reads

$$\left\{ \begin{array}{rclcl} \dfrac{\partial u}{\partial x} - \dfrac{1}{\gamma}\Delta u & = & f & \text{in} & \Omega; \\ u & = & 1 & \text{on} & \Gamma_1; \\ u & = & 0 & \text{on} & \Gamma_2; \\ \dfrac{\partial u}{\partial n} & = & 0 & \text{on} & \Gamma_3, \end{array} \right. \tag{18}$$

where

$$\Omega = (0,10)\times(0,1),\ \Gamma_1 = \{0\}\times(0,1),\ \Gamma_2 = \{10\}\times(0,1),\ \Gamma_3 = \partial\Omega\backslash(\Gamma_1\cup\Gamma_2),$$

In this problem Γ_1 and Γ_3 respectively are the inflow and outflow boundaries, as the advection velocity is $(1,0)$.

Numerical experiments : non-symmetric case

This behavior predicted by Theorem 4 is confirmed by our results. Table 3 displays the ratio between consecutive convergence rates. We consider $\Gamma = [\gamma_1, \gamma_2]$, $\gamma_1 = 10^{-2}$ and decreasing values of γ_2. Note that the minimum coercivity constant of the forms $a(\cdot, \cdot; \gamma)$ when $\gamma \in \Gamma$ is $\alpha = 1/\gamma_2$. We observe that the ratios are close to the theoretical asymptotic value of 4, getting closer as α increases (Theo.4).

α	Mode 1	ratio	Mode 2	ratio
0.064	402	-	520	-
0.128	1926	4.79	3474	6.53
0.256	8974	4.66	10954	4.36
0.512	43876	4.42	44886	4.10
1.024	173226	3.95	179069	3.99
2.048	690860	3.99	728603	4.07

Table: Convergence rates of PI Algorithm to compute the two first PGD modes for problem (18) vs α. The ratios are the quotient between two consecutive convergence rates.

Concluding remarks

- We have constructed an intrinsic tensorized approximation of parameterized elliptic equations (similar to PGD), with optimal approximation of each summand and orthogonality between residuals (similar to POD).
- Strong convergence of approximations.
- Very promising results for 2nd order elliptic equations, by power-iteration algorithm.