Efficient Solution Methodology Based on a Local Wave Tracking Strategy for High-Frequency Helmholtz Problems

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Abstract: The Helmholtz equation belongs to the classical equations of mathematical physics that are well understood from a mathematical view point. However, the numerical approximation of the solution is still a challenging problem in spite the tremendous progress made during the past fifty years. Indeed, the standard finite element method (FEM) is not well suited for solving Helmholtz problems in the mid- and high-frequency regime because of the quasi-optimality constant which grows with the wavenumber ka. In order to maintain a certain level of accuracy while increasing the frequency, a mesh refinement is required and/or higher order FEM are used, leading to a prohibitive computational cost for high wavenumbers. In response to this challenge, alternative techniques were proposed. Numerous of these approaches use the plane waves, since they are expected to better approximate highly oscillating waves. In the discontinuous Galerkin method (DGM), the solution is approximated at the element mesh level using a superposition of plane waves which results in a discontinuous solution along interior boundaries of the mesh. The continuity is then restored weakly with Lagrange multipliers. The rectangular and quadrilateral elements clearly outperform the standard Galerkin FEM. For example, for ka $\geq$ 10 and for a fixed level of accuracy, the so-called R-4-1 element reduces the total number of degrees of freedom (dofs) required by the Q1 finite element by a factor greater or equal to five. In spite of this impressive performance, the DGM has three important drawbacks. First, the method has to satisfy an inf-sup condition which is translated, in practice, as a compatibility requirement: the number of dofs of the Lagrange multiplier (corresponding to the dual variable) and of the field (the primal variable) cannot be chosen arbitrarily. The problem here is that there is no theoretical result on how to satisfy this compatibility requirement, except for the simple case of R-4-1 element. Hence, for other elements, the existing choices are based on numerical experiments only. The second major issue with the DGM is that it becomes unstable as we refine the mesh. Such instabilities occur because of the singularity of the local problems and, to some extent, to the loss of the linear independence of the plane waves as the step size mesh discretization tends to zero. The latter affects dramatically the stability of the global system due to its ill-conditioning nature. Finally, the DGM exhibits a loss of accuracy for unstructured mesh. We propose a new solution methodology for Helmholtz problems, that falls in the category of discontinuous Galerkin methods and least square approaches. The proposed formulation distinguishes itself from existing procedures by the well-posed character of the local problems and by the resulting global system which is associated with a positive semi-definite Hermitian matrix. More specifically, the computation domain is subdivided in quadrilateral- or triangular shaped elements. The solution is approximated, at the element level, by a superposition of adapted planewaves (that are solution of the Helmholtz equation). These adapted planewaves are choosen using a preprocessing based on an optimisation procedure.